CA23125 - The mETamaterial foRmalism approach to recognize cAncer (TETRA)
Working Groups 2, 2026

MATHEMATICAL MODELLING OF HIGHLY DISORDERED ANISOTROPIC STRUCTURES. PART 2.2. ENHANCED EFFECTIVE PERMITTIVITY MODEL TO INVESTIGATE ANISOTROPIC METAMATERIALS WITH HIGH DEGREE OF OF DISODER.

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Abstract

We develop a new method of complex potentials and constructive results on the $\mathbb{R}$-linear problem. The proposed method yields approximate and exact analytical formulas for the effective properties of dispersed composites with the strictly derived precision of their validity in concentration f and the contrast parameter ϱ for two-phase composites. First, we developed formulas for real-valued permittivity. Next, we explore complex values of the normalized permittivity of components ε_k for multi-phase composites. Consequently, the contrast parameters ϱ_k also become complex. Building upon the foundations laid out in this chapter, we extend these formulas to the complex domain through analytic continuation in terms of ϱ_k. To ensure the validity of this continuation, it is imperative that we establish a groundwork for understanding the interplay between complex permittivity and its matrix representation.

1. Implementation of Schwarz's method

1.1. **Explicit and implicit iterative schemes.** The method of successive approximations can be applied to equations of Schwarz's method by means of two different iterative schemes. The explicit iteration scheme has the form

(1) $$\varphi_k^{(0)}(z) = z,$$

(2) $$\varphi_k^{(p)}(z) = \sum_{m=1}^{N} \frac{\varrho_m}{2\pi i} \int_{L_m} \overline{\varphi_m^{(p-1)}(t)} E_1(t-z)\, \mathrm{d}t + z,\ z \in G_k,\ (p = 1, 2, \ldots).$$

The implicit iteration scheme to equations of Schwarz's method [12] with the same zero approximation (1) has the form

$$\varphi_k^{(p)}(z) - \tfrac{\varrho_k}{2\pi i}\int_{L_k} \overline{\varphi_k^{(p)}(t)} E_1(t-z)\mathrm{d}t = \tag{3}$$

$$\textstyle\sum_{m\neq k} \tfrac{\varrho_m}{2\pi i}\int_{L_m} \overline{\varphi_m^{(p-1)}(t)} E_1(t-z)\mathrm{d}t + z, \quad z \in G_k,\ (p = 1,2,\ldots).$$

The implicit scheme corresponds to equation of Schwarz's method.

In order to reduce computations, equations (1)-(3) are written without additive constants c_k. First, it is related to the observation that we need the derivative $\varphi_k'(z)$ to compute the effective constants. Second, an additive constant C in the approximation $\varphi_k^{(p-1)}(z)$ yields an additive constant in the next approximation $\varphi_k^{(p)}(z)$. This fact is established by the residue theorem

$$\frac{1}{2\pi i}\int_{L_m} \overline{C} E_1(t-z)\,\mathrm{d}t = \delta_{mk}\overline{C}, \quad z \in G_k, \tag{4}$$

where δ_{mk} stands for the Kronecker delta. Application of (1)-(3) leads to a power series in ϱ_k. This is the reason why the method is also called contrast expansion [9, 22, 5].

Let h_0 be given and h unknown functions from the space $\mathcal{H}(G_k)$ for a fixed k. The following integral equation has to be solved in every iteration step of (3)

$$h(z) - \frac{\varrho_k}{2\pi i}\int_{L_k} \overline{h(t)} E_1(t-z)\mathrm{d}t = h_0(z), \quad z \in G_k. \tag{5}$$

Introduce the compact integral operators in the space $\mathcal{H}(G_k)$

$$(P_k h)(z) = \frac{1}{2\pi i}\int_{L_k} \overline{h(t)} E_0(t-z)\mathrm{d}t \quad z \in G_k, \tag{6}$$

where the function $E_0(z)$ analytic in the periodicity cell Q is determined by its Taylor series

$$E_0(z) = E_1(z) - \frac{1}{z} = \sum_{k=1}^{\infty} S_{2k} z^{2k-1}. \tag{7}$$

Write equation (5) in the operator form

$$h - \varrho_k(\mathcal{S}h + P_k h) = h_0, \tag{8}$$

where

$$\mathcal{S}h(z) = \begin{cases} \frac{1}{2\pi i}\int_{L_k} \frac{\overline{h(t)}}{t-z}\mathrm{d}t, & z \in G_k, \\ \frac{1}{2}\overline{h(z)} + \frac{1}{2\pi i}\int_{L_k} \frac{\overline{h(t)}}{t-z}\mathrm{d}t, & z \in L_k. \end{cases} \tag{9}$$

Here, the boundary values of $\mathcal{S}h(z)$ on L_k are written in accordance with Sochocki's formulas

$$\lim_{\substack{\zeta\to z\\ \zeta\in G_k}} \frac{1}{2\pi i}\int_{L_k} \frac{\overline{h(t)}}{t-\zeta}\mathrm{d}t = \frac{1}{2}\overline{h(z)} + \frac{1}{2\pi i}\int_{L_k} \frac{\overline{h(t)}}{t-z}\mathrm{d}t,\ z\in L_k. \tag{10}$$

Consider the integral equation similar to (8)

$$h - \varrho_k \mathcal{S}h = h_0. \tag{11}$$

The singular operator $\mathcal{S}$ is bounded in the space $\mathcal{H}(L_k)$ of Hölder continuous functions on the curve L_k [23]. The space $\mathcal{H}(G_k)$ of functions analytically continued from L_k into G_k can be considered as a closed subspace of $\mathcal{H}(L_k)$. The following two theorems justify the approximation schemes developed in the next sections

Theorem 1 ([17])**.** *The operators $\mathcal{S}$ and $\mathcal{S}+P_k$ are bounded in $\mathcal{H}(G_k)$.*

Theorem 2 ([3])**.** *Equation (11) has the unique solution, which can be written by the inverse operator correctly defined and bounded in $\mathcal{H}(G_k)$ for $|\varrho_k|<1$*

$$h = (I-\varrho_k\mathcal{S})^{-1}h_0 = h_0 + \varrho_k\mathcal{S}h_0 + \varrho_k^2\mathcal{S}^2h_0 + \ldots. \tag{12}$$

An analogous assertion takes place for equation (5) [5].

The both methods (2) and (3) converge absolutely for any fixed $|\varrho_k|<1$ [15, 16]. In the case $|\varrho_k|=1$, the uniform convergence can be established by means of the regularization.

Equations (2) can be written in the form

$$\begin{aligned}&\varphi_k^{(p)}(z) = \textstyle\sum_{m=1}^N \frac{\varrho_m}{2\pi i}\int_{L_m}\overline{\varphi_m^{(p-1)}(t)}\frac{\mathrm{d}t}{t-z} + \sum_{m=1}^N \varrho_m\left(P_m\overline{\varphi_m^{(p-1)}}\right)(z) + z,\\ &z\in G_k,\ (p=1,2,\ldots).\end{aligned} \tag{13}$$

Equations (3) can be modified by splitting the integral term with pth iteration onto two terms with pth and $(p-1)$th iterations as follows

$$\begin{aligned}&\varphi_k^{(p)}(z) - \tfrac{\varrho_k}{2\pi i}\int_{L_k}\overline{\varphi_k^{(p)}(t)}\tfrac{\mathrm{d}t}{t-z} = \varrho_k\left(P_k\overline{\varphi_k^{(p-1)}}\right)(z)\\ &+\textstyle\sum_{m\neq k}\frac{\varrho_m}{2\pi i}\int_{L_m}\overline{\varphi_m^{(p-1)}(t)}E_1(t-z)\mathrm{d}t + z,\quad z\in G_k,\ (p=1,2,\ldots).\end{aligned} \tag{14}$$

The iterative scheme (14) can be treated as a mixed scheme when the self-induced charge of kth inclusions is partly included in the pth approximation and partly in the previous $(p-1)$th approximation.

Thus, Schwarz's method can be implemented by means of explicit and implicit iterative schemes with different modifications. The scheme (2) corresponds to the contrast expansion,

(3) to the expansion in the concentration of inclusions called cluster expansion in [22]. These two principal schemes with modifications like (14) have a lot of other names usually related to self-consistent methods. One can meet manipulations with the distorted balance of precision as in the difference scheme and Mori-Tanaka method.

For example, consider the approximation of (14) neglecting the terms with P_k for a two-phase composites ($\varrho_k = \varrho$)

$$\begin{aligned} &\varphi_k^{(p)}(z) - \varrho \tfrac{1}{2\pi i} \int_{L_k} \overline{\varphi_k^{(p)}(t)} \tfrac{\mathrm{d}t}{t-z} \approx \\ &\varrho \textstyle\sum_{m\neq k} \tfrac{1}{2\pi i} \int_{L_m} \overline{\varphi_m^{(p-1)}(t)} E_1(t-z)\mathrm{d}t + z, \quad z \in G_k,\ (p = 1,2,\ldots). \end{aligned} \tag{15}$$

Such an approximation can be accompanied by engineering arguments about the balance of flux produced by one inclusion and the flux by all the others. This is true up to $O(\varrho^2)$. Let the problem is solved in the first iteration. It can be solved analytically or numerically, it doesn't matter. But the problem is solved up to $O(\varrho^2)$. Now, let the second iteration is applied. The obtained second-order solution contains the term ϱ^2 analytically or numerically. The higher-order terms of f accompanying ϱ^2 can arise. And so forth. After a few iterations, one arrives at a formula or numerical data with a redundant tail of high-order terms in ϱ and f.

Equations (15) can be ”improved” by the approximation of integral operator $\varrho \left(P_k \overline{\varphi_k^{(p-1)}} \right)(z) \approx \varrho C_k$ for some constants C_k. Such a type of approximation was proposed for elastic composites in [20]. It was based on the integral approximation, hence, was of order $O(f)$. This approximation was used in [7] for a square array of circular inclusions. Though the applied method gave the proper result of order $O(f)$, formulas for the effective constants were written with the tails up to $O(f^4)$.

Other schemes based on various mathematical manipulations, such as Brugemann's differential scheme, can be considered as a separate method. However, if one examines the result within the obtained precision, it becomes clear that the result coincides with the dilute Clausius-Mossotti (Maxwell) approximation written in another asymptotic form. This question will be discussed below.

1.2. **Contrast expansion (explicit scheme).** In the present section, we pay attention to the explicit iteration scheme One can see that every iteration (2) increases the contrast parameters precision by the multiplier of order $O(|\varrho|)$. The parameter ϱ is used with the modulus in order to avoid confusion in other places when $\varrho = \varrho_k$ for a two-phase composite. The integration operator increases the precision by $O(f)$, since

$$F \mapsto |G_k| \left(\frac{1}{|G_k|} \int_{G_k} F \mathrm{d}x_1 \mathrm{d}x_2 \right), \tag{16}$$

where the operator in the parentheses is bounded in $\mathcal{H}(G_k)$. The multiplier $|G_k|$ is of order f. Therefore,

$$\frac{d\varphi_k}{dz}(z) = \frac{d\varphi_k^{(p)}}{dz}(z) + O(|\varrho|^{p+1} f^p), \quad z \in G_k \cup L_k. \tag{17}$$

Integrating this relation we estimate the integral

$$\int_{G_k} \frac{d\varphi_k}{dz}(z)\, \mathrm{d}x_1 \mathrm{d}x_2 = \int_{G_k} \frac{d\varphi_k^{(p)}}{dz}(z)\, \mathrm{d}x_1 \mathrm{d}x_2 + O(|\varrho f|^{p+1}). \tag{18}$$

This integral will be applied to determine the effective constants. The multiplier ϱ_m in (2) guarantees the required precision in ϱ. We will use the expansion of the kernel $E_1(t-z)$ in terms of order $f^{1/2}$ that requires a more careful study of precision in concentration. Hereafter in this section, the clear behavior of precision in ϱ at every step is omitted for shortness.

We now proceed to develop a symbolic algorithm for the effective constants for a given p. It follows from (2) that the functions $\varphi_k^{(q)}(z)$ for $q = p-1$ have to be determined up to $O(f^{q+1/2})$. The kernel $E_1(t-z)$ in the integral from (2) for $\varphi_k^{(q)}(z)$ has to be estimated with the same precision.

Introduce the integral frequently met in the theory of analytic functions [23]

$$J_k(z) = \frac{1}{2\pi i} \int_{L_k} \frac{\overline{t}}{t-z} \mathrm{d}t, \quad z \in G_k. \tag{19}$$

This function is analytic in G_k and Hölder continuous in $G_k \cup L_k$ [19, 23]. It is worth noting that the limit values of $J_k(z)$ as $z \to L_k$ after the application of Sochocki's formula can be written by means of the singular integral

$$J_k(\tau) = \frac{\overline{\tau}}{2} + \frac{1}{2\pi i} \int_{L_k} \frac{\overline{t}}{t-\tau} \mathrm{d}t, \quad \tau \in L_k. \tag{20}$$

The iterations (13) include the integral

$$I_{mk}^{(q)}(z) = \frac{1}{2\pi i} \int_{L_m} \overline{\varphi_m^{(q)}(t)}\, E_1(t-z)\, \mathrm{d}t. \tag{21}$$

It follows from $\varphi_m^{(0)}(t) = t$ and (7) that the singular part of the integral $I_{kk}^{(0)}(z)$ coincides with $J_k(z)$. The integral (21) will be estimated below in two cases.

i) First, it is assumed that $m \neq k$. Consider the simple expression

$$t - z = (t - a_m) + (a_m - a_k) - (z - a_k), \tag{22}$$

where $|a_m - a_k|$ is the dominating term in comparison with $|t - a_m|$ and $|z - a_k|$. The value $|t - a_m + a_k - z|$ is of order $O(r) = O(f^{1/2})$. The Taylor approximation of $E_1(t - z)$ holds near $a_m - a_k$

$$E_1(t-z) = \sum_{n=0}^{2q+1} (-1)^n E_{n+1}(a_m - a_k)(t - a_m + a_k - z)^n + O(f^{q+1}). \tag{23}$$

Using (23) we estimate the integral (21)

$$I_{mk}^{(q)}(z) = \sum_{n=0}^{2q+1} (-1)^n E_{n+1}(a_m - a_k)\frac{1}{2\pi i}\int_{L_m} \overline{\varphi_m^{(q)}(t)}(t - a_m + a_k - z)^n \mathrm{d}t + O(f^{q+1}). \tag{24}$$

ii) Consider now the case $m = k$. Using (7) we approximate $E_1(t - z)$ in G_k by expression

$$E_1(t-z) = \frac{1}{t-z} - \sum_{n=1}^{q+1} S_{2n}(t-z)^{2n-1} + O(f^{q+1}). \tag{25}$$

Substitution of (25) into (21) yields

$$I_{kk}^{(q)}(z) = \frac{1}{2\pi i}\int_{L_k} \overline{\varphi_k^{(q)}(t)}\frac{\mathrm{d}t}{t-z} - \sum_{n=1}^{q+1} \frac{S_{2n}}{2\pi i}\int_{L_k} \overline{\varphi_k^{(q)}(t)}(t-z)^{2n-1}\mathrm{d}t + O(f^{q+1}). \tag{26}$$

The asymptotic formulas (24) and (26) yield the estimation for

$$\varphi_k^{(q)}(z) = z + \sum_{m=1}^{N} \varrho_m I_{mk}^{(q-1)}(z). \tag{27}$$

1.3. **Second iteration in contrast expansion.** The general method described above can be implemented in symbolic form and simplified within a fixed precision. We illustrate the general scheme for $p = 2$. Even such a low-order approximation yields a new analytical approximation for the effective permittivity tensor. We will use the following approximation

$$\begin{aligned} \varepsilon_{11} &= 1 + 2\mathrm{Re}\ \textstyle\sum_{k=1}^{N} \varrho_k \int_{G_k} \frac{d}{dz}\varphi_k^{(2)}(\xi_1 + i\xi_2)\ \mathrm{d}\xi_1 \mathrm{d}\xi_2 + O(|\varrho|^3 f^{7/2}), \\ \varepsilon_{12} &= -2\mathrm{Im}\ \textstyle\sum_{k=1}^{N} \varrho_k \int_{G_k} \frac{d}{dz}\varphi_k^{(2)}(\xi_1 + i\xi_2)\ \mathrm{d}\xi_1 \mathrm{d}\xi_2 + O(|\varrho|^3 f^{7/2}). \end{aligned} \tag{28}$$

It will be seen below that it is sufficient to take the approximation $\frac{d}{dz}\varphi_k^{(2)}$ up to $O(|\varrho|^2 f^{5/2})$ in order to reach the specified in (28) precision.

First, investigate the precision in ϱ. The idea can be easily represented for a two-phase composite when the constant $\varrho = \varrho_k$ for all $k = 1, 2, \ldots, N$. The iterative scheme (2) yields $\varphi_k^{(0)} = \alpha_0$ and $\varphi_k^{(1)} = \alpha_1 \varrho + \beta_1$, where the terms α_0, α_1, β_1 are constants in ϱ. The second iteration can be schematically written as $\varphi_k^{(2)} = \varrho(\alpha_1 \varrho + \beta_1) + \alpha_0$. After substitution into (28), one can see that one may neglect the coefficient α_1. Therefore, it is sufficient to determine $\varphi_k^{(1)}$ up to $O(1)$, as $\varrho \to 0$,

$$\varphi_k^{(1)}(z) = \sum_{m=1}^{N} \frac{\varrho_m}{2\pi i} \int_{L_m} \bar{t}\, E_1(t-z)\, \mathrm{d}t + z = z + O(1), \quad z \in G_k. \tag{29}$$

For completeness, we have to note that the following singular integral is a bounded function in $G_k \cup L_k$

$$\frac{1}{2\pi i} \int_{L_k} \bar{t}\, E_1(t-z)\, \mathrm{d}t = J_k(z) - \frac{1}{2\pi i} \int_{L_k} \bar{t}\, E_0(t-z)\, \mathrm{d}t. \tag{30}$$

Here, equations (7) and (19) are used. The second approximation has the form

$$\varphi_k^{(2)}(z) = \sum_{m=1}^{N} \frac{\varrho_m}{2\pi i} \int_{L_m} \bar{t}\, E_1(t-z)\, \mathrm{d}t + z + O(|\varrho|^2), \quad z \in G_k. \tag{31}$$

Below, we omit the precision in ϱ for shortness and keep track of the precision in f. Differentiate equation (31)

$$\frac{d}{dz}\varphi_k^{(2)}(z) = 1 + \frac{1}{\pi} \sum_{m=1}^{N} \varrho_m \frac{d}{dz} g_m(z), \tag{32}$$

where the analytic in G_m function is introduced

$$g_m(z) = \frac{1}{2i} \int_{L_m} \bar{t} E_1(t-z)\mathrm{d}t, \quad z \in G_k. \tag{33}$$

The complex Green formulas hold for a function $w(z,\bar{z})$ continuously differentiable in a smooth closed domain G [23]

$$\int_G \frac{\partial w}{\partial \bar{z}}\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 = \frac{1}{2i} \int_{\partial G} w\, \mathrm{d}t, \quad \int_G \frac{\partial w}{\partial z}\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 = -\frac{1}{2i} \int_{\partial G} w\, \mathrm{d}\bar{t}. \tag{34}$$

Introduce the averaged contrast parameter over inclusions

$$\langle \varrho \rangle = \sum_{k=1}^{N} \varrho_k |G_k|. \tag{35}$$

Using (32) and (28) we obtain

$$\begin{aligned} \varepsilon_{11} &= 1 + 2\langle \varrho \rangle + \mathrm{Re}\, \tfrac{2}{\pi} \textstyle\sum_{k=1}^{N} \sum_{m=1}^{N} \varrho_k \varrho_m h_{km} + O(|\varrho|^3 f^{7/2}), \\ \varepsilon_{12} &= -\mathrm{Im}\, \tfrac{2}{\pi} \textstyle\sum_{k=1}^{N} \sum_{m=1}^{N} \varrho_k \varrho_m h_{km} + O(|\varrho|^3 f^{7/2}). \end{aligned} \tag{36}$$

where

$$h_{km} = \int_{G_k} \frac{dg_m}{dz}(z)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 = -\frac{1}{2i} \int_{L_k} \bar{t}\, g_m(t)\, \mathrm{d}t \tag{37}$$

by the second Green formula (34).

i) Consider the integral (33) when $m \neq k$. Introduce for shortness the designations $E_{p,mk} = E_p(a_m - a_k)$ and $T = t - a_m$, $Z = z - a_k$. We represent the function $E_1(t-z)$ by the Taylor formula

$$E_1(t-z) = E_{1,mk} - E_{2,mk}(T-Z) + E_{3,mk}(T-Z)^2 - E_{4,mk}(T-Z)^3 + E_{5,mk}(T-Z)^4 + O(f^{5/2}). \tag{38}$$

Substitute (38) into (33)

$$g_m(z) = \frac{1}{2i}\int_{L_m} \bar{t}\,[f_1(z) + f_2(z,t)]\,\mathrm{d}t + O(f^{5/2}), \tag{39}$$

where the following designations are introduced

$$\begin{aligned} f_1(z) &= E_{1,mk} + E_{2,mk}Z + E_{3,mk}Z^2 + E_{4,mk}Z^3 + E_{5,mk}Z^4, \\ f_2(z,t) &= -(E_{2,mk} + 2ZE_{3,mk} + 3Z^2E_{4,mk} + 4E_{5,mk}Z^3)T \\ &+ (E_{3,mk} + 3E_{4,mk}Z + 6E_{5,mk}Z^2)T^2 - (E_{4,mk} + 4E_{5,mk}Z)T^3 + E_{5,mk}T^4. \end{aligned} \tag{40}$$

Remark 1. *The result (40) and others below are obtained due to symbolic computations with the package Mathematica®. The corresponding symbolic computations can increase the precision* [4, Chapter 2].

Using the first Green formula (34) introduce the static complex moments of the domain G_m

$$\begin{aligned} s_{qm} &= \tfrac{1}{2i}\int_{L_m} \bar{t}(t-a_m)^q \mathrm{d}t = \int_{G_m} (t-a_m)^q \mathrm{d}x_1\mathrm{d}x_2 = \\ &\tfrac{1}{2i}\int_{L_m} (\overline{t-a_m})(t-a_m)^q \mathrm{d}t \quad (q = 0,1,\ldots) \end{aligned} \tag{41}$$

and calculate

$$s_{0m} = \frac{1}{2i}\int_{L_m} \bar{t}\,\mathrm{d}t = \int_{G_m} \mathrm{d}x_1\mathrm{d}x_2 = |G_m|. \tag{42}$$

The dimensionless complex static moments can be introduced as follows

$$s_{qm}^{(0)} = s_{qm}s_{0m}^{-\frac{q}{2}-1}. \tag{43}$$

In particular, (43) yields $s_{0m}^{(0)} = 1$. It follows from (41) and (43) that

$$s_{qm} = O(f^{\frac{q}{2}+1}), \quad s_{qm}^{(0)} = O(1). \tag{44}$$

Continue to use the designations

$$E_{p,mk} = E_p(a_m - a_k), \quad p = 2,3,\ldots; \quad m,k = 1,2,\ldots,N, \tag{45}$$

including the case $m = k$. Here, it is assumed for shortness that $E_p(a_m - a_k) = S_p$ when a_m coincides with a_k. The lattice sums S_p are introduced in [11]. Then, equation (39) can be written in the form

$$
\begin{aligned}
g_m(z) = f_1(z)s_{0m} - (E_{2,mk} + 2ZE_{3,mk} + 3Z^2E_{4,mk} + 4Z^3E_{5,mk})s_{1m} \\
+(E_{3,mk} + 3E_{4,mk}Z + 6E_{5,mk}Z^2)s_{2m} \\
-(E_{4,mk} + 4E_{5,mk}Z)s_{3m} + E_{5,mk}s_{4m} + O(f^{7/2}),
\end{aligned}
\tag{46}
$$

where the estimation (44) is used and $Z = z - a_k$. Calculate

$$
\begin{aligned}
\frac{dg_m}{dz}(z) = (E_{2,mk} + 2E_{3,mk}Z + 3E_{4,mk}Z^2 + 4E_{5,mk}Z^3)s_{0m} \\
-2(E_{3,mk} + 3ZE_{4,mk} + 6Z^2E_{5,mk})s_{1m} + 3(E_{4,mk} + 4E_{5,mk}Z)s_{2m} \\
-4E_{5,mk}s_{3m} + O(f^3).
\end{aligned}
\tag{47}
$$

The differentiation operator decreases the precision on $f^{1/2}$, since $|z - a_k| = O(f^{1/2})$. Using the first Green formula (34), calculate the integral (37) with $\frac{dg_m}{dz}(z)$ given by the approximation (47)

$$
\begin{aligned}
h_{km} = (E_{2,mk}s_{0k} + 2E_{3,mk}s_{1k} + 3E_{4,mk}s_{2k} + 4E_{5,mk}s_{3k})s_{0m} \\
-2(E_{3,mk}s_{0k} + 3E_{4,mk}s_{1k} + 6E_{5,mk}s_{2k})s_{1m} \\
+3(E_{4,mk}s_{0k} + 4E_{5,mk}s_{1k})s_{2m} - 4E_{5,mk}s_{0k}s_{3m} + O(f^4).
\end{aligned}
\tag{48}
$$

Write (48) in the form

$$
\begin{aligned}
h_{km} = E_{2,mk}s_{0k}s_{0m} + 2E_{3,mk}(s_{1k}s_{0m} - s_{1m}s_{0k}) \\
+3E_{4,mk}(s_{2k}s_{0m} + s_{2m}s_{0k} - 2s_{1m}s_{1k}) \\
+4E_{5,mk}[s_{3k}s_{0m} - s_{0k}s_{3m} + 3(s_{1k}s_{2m} - s_{2k}s_{1m})] + O(f^4).
\end{aligned}
\tag{49}
$$

ii) Consider the integral (33) when $m = k$. It is equal to the integral (30)

$$
g_k(z) = J_k(z) - \frac{1}{2i}\int_{L_k} \bar{t}\, E_0(t - z)\, \mathrm{d}t. \tag{50}
$$

We find for the considered periodic square unit cell

$$
E_0'(t - z) = \pi + 3S_4(t - z)^2 + 7S_8(t - z)^6 + O(f^{7/2}). \tag{51}
$$

Substitute (51) into (50) and use the first formula (34)

$$
g_k'(z) = J_k'(z) + \pi s_{0k} + 3S_4 \int_{G_k} (t - z)^2\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 + O(f^{5/2}). \tag{52}
$$

The term $(t - z)^6$ will be neglected due to the considered precision.

Calculate the integral from (52)

$$
\int_{G_k} [t - a_k - (z - a_k)]^2\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 = s_{2k} - 2s_{1k}(z - a_k) + s_{0k}(z - a_k)^2. \tag{53}
$$

Then, h_{kk} defined by (37) can be calculated by the asymptotic formula

$$h_{kk} = \int_{G_k} J_k'(z)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 + \pi s_{0k}^2 + 6S_4(s_{2k}s_{0k} - s_{1k}^2) + O(f^4). \tag{54}$$

Therefore, the values h_{km} can be found by (48) and (54).

Introduce the value

$$\mathcal{J} = \sum_{k=1}^{N} \varrho_k^2 \int_{G_k} J_k'(z)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 = -\frac{1}{2\mathrm{i}} \sum_{k=1}^{N} \varrho_k^2 \int_{L_k} J_k(t)\, \mathrm{d}\bar{t}. \tag{55}$$

Applying Sochocki's formulas to $J_k(z)$ defined by (19) and substituting the result into (55) we obtain

$$\mathcal{J} = \frac{1}{4\pi} \sum_{k=1}^{N} \varrho_k^2 \int_{L_k} \mathrm{d}\bar{t} \int_{L_k} \frac{\overline{\tau}}{\tau - t}\, \mathrm{d}\tau. \tag{56}$$

We now proceed to calculate the double sum $\frac{1}{\pi}\sum_{k=1}^{N}\sum_{m=1}^{N} \varrho_k \varrho_m h_{km}$ using the asymptotic formulas (49) and (54). It is convenient to split this sum into the following ones. Introduce the double sums

$$\mathcal{L} = \frac{1}{\pi} \sum_{k,m=1}^{N} \varrho_k \varrho_m |G_k||G_m| E_{2,mk} \tag{57}$$

and

$$\mathcal{V}(1) = \frac{2}{\pi} \sum_{k,m=1}^{N} \varrho_k \varrho_m |G_k||G_m| E_{3,mk} (s_{1k}^{(0)} |G_k|^{\frac{1}{2}} - s_{1m}^{(0)} |G_m|^{\frac{1}{2}}) \tag{58}$$

using the designation (45) and the normalized static moments (43). Analogously introduce

$$\begin{aligned}\mathcal{V}(2) = \tfrac{3}{\pi} \textstyle\sum_{k,m=1}^{N} \varrho_k \varrho_m |G_k||G_m| E_{4,mk} \times \\ \left(|G_k| s_{2k}^{(0)} + |G_m| s_{2m}^{(0)} - 2\left(|G_k||G_m|\right)^{\frac{1}{2}} s_{1m}^{(0)} s_{1k}^{(0)} \right)\end{aligned} \tag{59}$$

and

$$\begin{aligned}\mathcal{V}(3) = \tfrac{4}{\pi} \textstyle\sum_{k,m=1}^{N} \varrho_k \varrho_m |G_k||G_m| E_{5,mk} \times \\ \left[|G_k|^{\frac{3}{2}} s_{3k}^{(0)} - |G_m|^{\frac{3}{2}} s_{3m}^{(0)} + 3\left(|G_k|^{\frac{1}{2}} |G_m| s_{1k}^{(0)} s_{2m}^{(0)} - |G_m|^{\frac{1}{2}} |G_k| s_{2k}^{(0)} s_{1m}^{(0)} \right) \right].\end{aligned} \tag{60}$$

Put

$$\mathcal{V} = \mathcal{V}(1) + \mathcal{V}(2) + \mathcal{V}(3). \tag{61}$$

Then, equation (36) can be written in the form

$$\begin{aligned}\varepsilon_{11} &= 1 + 2\langle \varrho \rangle + 2\mathrm{Re}\ (\mathcal{J} + \mathcal{L} + \mathcal{V}) + O(|\varrho|^3 f^4), \\ \varepsilon_{12} &= -2\mathrm{Im}\ (\mathcal{J} + \mathcal{L} + \mathcal{V}) + O(|\varrho|^3 f^4).\end{aligned} \tag{62}$$

We again write formulas with the explicitly shown precision in ϱ. The asymptotic formula (62) is a result of the second iteration in the contrast expansion. The value $|G_k|$ can be considered as the volume fraction of the component having the contrast parameter ϱ_k.

Recall that the periodicity cell Q has the unit area, which means normalizing the centers a_k and using the scaled to the unit periodicity cell for the Eisenstein functions. Hence, the terms $\mathcal{L}$, $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ are of orders $O(f^2)$, $O(f^{5/2})$, $O(f^3)$ and $O(f^{7/2})$, respectively. The term $\mathcal{J}$ from (55) has the order $O(f)$ except at some shapes when it is equal to zero. Therefore, one can introduce the dimensionless cumulative factors

$$\mathcal{J}_0 = f^{-1}\mathcal{J}, \quad \mathcal{L}_0 = f^{-2}\mathcal{L}, \quad \mathcal{V}_{0j} = f^{-\frac{j+4}{2}}\mathcal{V}_j. \tag{63}$$

Then, (62) can be written up to $O(|\varrho|^3 f^4)$ in the form

$$\begin{aligned} \varepsilon_{11} &= 1 + 2\langle\varrho\rangle + 2\mathrm{Re}\,\left[f\mathcal{J}_0 + f^2\mathcal{L}_0 + f^{5/2}\left(\mathcal{V}_{01} + f^{1/2}\mathcal{V}_{02} + f\mathcal{V}_{03}\right)\right], \\ \varepsilon_{12} &= -2\mathrm{Im}\,\left[f\mathcal{J}_0 + f^2\mathcal{L}_0 + f^{5/2}\left(V_{01} + f^{1/2}\mathcal{V}_{02} + f\mathcal{V}_{03}\right)\right]. \end{aligned} \tag{64}$$

In the case of circular inclusions $J_k(z) = 0$, see [5], and all the static moments s_{qm} vanishes for $q \geq 1$. Hence, the terms $\mathcal{J}$ and $\mathcal{V}$ vanish.

In order to calculate the components ε_{22} and $\varepsilon_{21} = \varepsilon_{12}$ one can rotate the structure about the angle $\frac{\pi}{2}$ and apply the same method. Let the inclusion G_k be transformed into G_k^* after this rotation. Then, (62) yields

$$\begin{aligned} \varepsilon_{22} &= 1 + 2\langle\varrho\rangle + 2\mathrm{Re}\,\left(\mathcal{J}^* + \mathcal{L}^* + \mathcal{V}^*\right) + O(|\varrho|^3 f^4), \\ \varepsilon_{21} &= \varepsilon_{12} = 2\mathrm{Im}\,\left(\mathcal{J}^* + \mathcal{L}^* + \mathcal{V}^*\right) + O(|\varrho|^3 f^4), \end{aligned} \tag{65}$$

where the values with the asterisk correspond to the values from (62). The expressions for ε_{12} in (64) and ε_{21} in (65) have the opposite signs since after the rotation, the coordinate system x_1Ox_2 changes its orientation.

Let $\mathcal{J}$ be given by (56). Then,

$$\mathcal{J}^* = \frac{1}{4\pi}\sum_{k=1}^{N}\varrho_k^2\int_{L_k^*} \mathrm{d}\overline{t^*}\int_{L_k^*}\frac{\overline{\tau^*}}{\tau^* - t^*}\,\mathrm{d}\tau^* = -\mathcal{J}, \tag{66}$$

The relation (66) is established by the change of variables $t^* = \mathrm{i}t$ and $\tau^* = \mathrm{i}\tau$ in the integral (66).

The next term is calculated below

$$\mathcal{L}^* = \frac{1}{\pi}\sum_{m,k=1}^{N}\varrho_k\varrho_m|G_k||G_m|E_2^*(i(a_k - a_k)) = 2\langle\varrho\rangle^2 - \mathcal{L}. \tag{67}$$

One can check that the static moments (41) are transformed after rotation in the following way

$$s_{qm}^{*} = \mathrm{i}^{q} s_{qm} \quad (q = 1, 2, \ldots). \tag{68}$$

The same relation (68) holds for the normalized static moments $s_{qm}^{(0)*}$. Introduce the value

$$\mathcal{V}_1^{*} = \frac{2}{\pi} \sum_{k,m=1}^{N} \varrho_k \varrho_m |G_k||G_m| E_3(a_k^{*} - a_k^{*})(s_{1k}^{(0)*}|G_k|^{\frac{1}{2}} - s_{1m}^{(0)*}|G_m|^{\frac{1}{2}}). \tag{69}$$

One can see that $\mathcal{V}_1^{*} = -\mathcal{V}_1$. The same arguments yield $\mathcal{V}_2^{*} = -\mathcal{V}_2$ and $\mathcal{V}_3^{*} = -\mathcal{V}_3$, hence, $\mathcal{V}^{*} = -\mathcal{V}$.

Let I denote the unit tensor. The effective tensor can be calculated by (64) and (65) up to $O(|\varrho|^3 f^4)$

$$\begin{aligned} \boldsymbol{\varepsilon}_{\perp} = {} & (1 + 2\langle\varrho\rangle) I + 2f \begin{pmatrix} \mathrm{Re}\, \mathcal{J}_0 & -\mathrm{Im}\, \mathcal{J}_0 \\ -\mathrm{Im}\, \mathcal{J}_0 & -\mathrm{Re}\, \mathcal{J}_0 \end{pmatrix} \\ & + 2f^2 \begin{pmatrix} \mathrm{Re}\, \mathcal{L}_0 & -\mathrm{Im}\, \mathcal{L}_0 \\ -\mathrm{Im}\, \mathcal{L}_0 & 2 - \mathrm{Re}\, \mathcal{L}_0 \end{pmatrix} + 2f^{5/2} \begin{pmatrix} \mathrm{Re}\, \mathcal{V}_{01} & -\mathrm{Im}\, \mathcal{V}_{01} \\ -\mathrm{Im}\, \mathcal{V}_{01} & -\mathrm{Re}\, \mathcal{V}_{01} \end{pmatrix} \\ & + 2f^3 \begin{pmatrix} \mathrm{Re}\, \mathcal{V}_{02} & -\mathrm{Im}\, \mathcal{V}_{02} \\ -\mathrm{Im}\, \mathcal{V}_{02} & -\mathrm{Re}\, \mathcal{V}_{02} \end{pmatrix} + 2f^{7/2} \begin{pmatrix} \mathrm{Re}\, \mathcal{V}_{03} & -\mathrm{Im}\, \mathcal{V}_{03} \\ -\mathrm{Im}\, \mathcal{V}_{03} & -\mathrm{Re}\, \mathcal{V}_{03} \end{pmatrix}. \end{aligned} \tag{70}$$

The value $\mathcal{J}$ defined by (56) can be considered as the sum of cumulative shape factors of inclusions G_k. The value $\mathcal{L}$ defined by (58) represents the mutual locations of inclusions. Therefore, $\mathcal{L}$ can be considered as the cumulative location factor. The term $\mathcal{V}$ defined by (58)-(61) can be considered as the cumulative mixed location-shape factor. It consists of the terms of orders $O(f^{5/2})$, $O(f^3)$ and $O(f^{7/2})$ which may vanish.

The formula (70) can be easily implemented. For real permittivity, one can find numerical examples in [14] not accessible by FEM because of computational restrictions.

1.4. **Two-phase composite.** Consider a two-phase composite with the same contrast parameter $\varrho = \varrho_k$. Then, $\langle\varrho\rangle$ becomes ϱf. Following [14] introduce the parameters $\mathcal{J}'$, $\mathcal{L}'$, $\mathcal{V}'(j)$ and $\mathcal{V}'$ equal to $\mathcal{J}$, $\mathcal{L}$,$\mathcal{V}(j)$ and $\mathcal{V}$, respectively, after the substitution $\varrho_k = 1$ into (56)-(61). For instance,

$$\mathcal{L}' = \frac{1}{\pi} \sum_{k,m=1}^{N} |G_k||G_m| E_{2,mk}. \tag{71}$$

In the considered case, (70) becomes

$$\boldsymbol{\varepsilon}_{\perp} = (1+2\varrho f)I + 2f\varrho^2 \begin{pmatrix} \operatorname{Re}\mathcal{J}_0' & -\operatorname{Im}\mathcal{J}_0' \\ -\operatorname{Im}\mathcal{J}_0' & -\operatorname{Re}\mathcal{J}_0' \end{pmatrix}$$

$$+2\varrho^2 f^2 \begin{pmatrix} \operatorname{Re}\mathcal{L}_0' & -\operatorname{Im}\mathcal{L}_0' \\ -\operatorname{Im}\mathcal{L}_0' & 2-\operatorname{Re}\mathcal{L}_0' \end{pmatrix} + 2\varrho^2 f^{5/2} \begin{pmatrix} \operatorname{Re}\mathcal{V}_{01}' & -\operatorname{Im}\mathcal{V}_{01}' \\ -\operatorname{Im}\mathcal{V}_{01}' & -\operatorname{Re}\mathcal{V}_{01}' \end{pmatrix} \tag{72}$$

$$+2\varrho^2 f^3 \begin{pmatrix} \operatorname{Re}\mathcal{V}_{02}' & -\operatorname{Im}\mathcal{V}_{02}' \\ -\operatorname{Im}\mathcal{V}_{02}' & -\operatorname{Re}\mathcal{V}_{02}' \end{pmatrix} + 2\varrho^2 f^{7/2} \begin{pmatrix} \operatorname{Re}\mathcal{V}_{03}' & -\operatorname{Im}\mathcal{V}_{03}' \\ -\operatorname{Im}\mathcal{V}_{03}' & -\operatorname{Re}\mathcal{V}_{03}' \end{pmatrix}.$$

The component ε_{22} in the formula (91) from [14] contains an erroneous sign in $\mathcal{J}$ and $\mathcal{V}$ corrected here in (72).

The first two terms of (72) deserve special attention. The following universal formulas hold for any 2D two-phase dispersed composite

$$\begin{aligned} \boldsymbol{\varepsilon}_{\perp} &= (1+2\varrho f)I + O(\varrho^2), \\ \boldsymbol{\varepsilon}_{\perp} &= (1+2\varrho f)I + 2f\varrho^2 \begin{pmatrix} \operatorname{Re}\mathcal{J}_0' & -\operatorname{Im}\mathcal{J}_0' \\ -\operatorname{Im}\mathcal{J}_0' & -\operatorname{Re}\mathcal{J}_0' \end{pmatrix} + O(f^2). \end{aligned} \tag{73}$$

The first formula (73) can be independently justified by the Wiener bounds [24] written for two-phase composites

$$\left[(f\varepsilon_1^{-1} + (1-f)\varepsilon^{-1})\right]^{-1} \le \varepsilon_e \le f\varepsilon_1 + (1-f)\varepsilon. \tag{74}$$

Here, the phases have the permittivity ε_1 and ε, the concentrations f and $1-f$, respectively. We now consider the normalized positive permittivity when $\varepsilon = 1$. Substitution of $\varepsilon_1 = \frac{1+\varrho}{1-\varrho}$ and $\varepsilon = 1$ into (74) yields the lower and upper bounds coinciding up to $O(\varrho^2)$.

The term from the second formula (73) can be determined by the sum

$$\mathcal{J}_0' = \sum_{k=1}^{N} j_k, \tag{75}$$

where j_k means the dimensionless shape factor of kth inclusion

$$j_k = \frac{1}{4\pi f} \int_{L_k} \mathrm{d}\bar{t} \int_{L_k} \frac{\overline{\tau}}{\tau - t}\, \mathrm{d}\tau. \tag{76}$$

One can see that j_k is not changed under a translation $t \mapsto t + a$ since

$$j_k(a) - j_k(0) = \frac{1}{4\pi f} \int_{L_k} \mathrm{d}\bar{t} \int_{L_k} \frac{\overline{a}}{\tau - t}\, \mathrm{d}\tau = \frac{1}{4\pi f} \int_{L_k} \mathrm{d}\bar{t}\, \overline{a}\, \pi \mathrm{i} = 0. \tag{77}$$

The integral (76) is not changed under a scaling $t \mapsto ct$ since $f \mapsto c^2 f$.

We now proceed to consider a two-phase macroscopically isotropic dispersed composite. The tensor $\boldsymbol{\varepsilon}_\perp = \varepsilon_e I$ is expressed through the scalar value ε_e called the effective permittivity. Due to the properties of analytic functions, such a relation can take place for almost all ϱ and f, if the following conditions are fulfilled

(78) $$\mathcal{J}_0' = \mathcal{V}_0' = 0, \quad \mathcal{L}_0' = 1.$$

In this case, (72) yields the ClaudiusMossotti approximation with the exactly written precision in concentration and contrast parameter

(79) $$\varepsilon_e = 1 + 2\varrho f + 2\varrho^2 f^2 + O(|\varrho|^3 f^4) = \frac{1+\varrho f}{1-\varrho f} + O(|\varrho|^3 f^4).$$

We now end up analyzing the results obtained from the explicit scheme for arbitrary shapes of inclusions with real permittivity. These results will be developed for complex permittivity and circular shape.

1.5. **First iteration of the implicit scheme for a two-phase composite.** In the present section, we analyze the precision for the first iteration of implicit scheme

(80) $$\begin{aligned} &\varphi_k^{(1)}(z) - \tfrac{\varrho}{2\pi i}\textstyle\int_{L_k} \overline{\varphi_k^{(1)}(t)} E_1(t-z)\mathrm{d}t = \sum_{m\neq k} \tfrac{\varrho}{2\pi i}\int_{L_m} \bar{t} E_1(t-z)\mathrm{d}t + z. \\ &z \in G_k. \end{aligned}$$

For simplicity, a two-phase composite is considered when $\varrho = \varrho_k$ for all $(k = 1, 2, \ldots, N)$. We are interested in the effective constant calculated up to $O(f^3)$ for an arbitrary ϱ. We now demonstrate that it is sufficient to determine $\varphi_k^{(1)}(z)$ up to $O(f^2)$.

Theorem 3. *Any approximation* $\varphi_k^{(p)}(z)$ *of implicit scheme* (3) *coincides with* $\varphi_k^{(1)}(z)$ *up to* $O(f^2)$.

Proof. Equation (80) can be simplified up to $O(f^2)$ to equation

(81) $$\varphi_k^{(1)}(z) - \frac{\varrho}{2\pi i}\int_{L_k} \overline{\varphi_k^{(1)}(t)} E_1(t-z)\mathrm{d}t = z, \quad z \in G_k$$

due to the boundness of the integral operator established in Theorem 1. Here, we use the asymptotic formula $E_1(t-z) = E_1(a_m - a_k) + O(f^{1/2})$, as $f \to 0$, for dispersed composites. It is worth noting that the integral operator over L_k increases the precision by the multiplier f. The constant term $E_1(a_m - a_k)$ produces a constant that does not impact the final result, hence it can be discarded. Moreover, in order to find $\varphi_k^{(1)}(z)$ up to $O(f^2)$ it is sufficient instead of (81) to solve equation

(82) $$\varphi_k^{(1)}(z) - \frac{\varrho}{2\pi i}\int_{L_k} \frac{\overline{\varphi_k^{(1)}(t)}}{t-z}\mathrm{d}t = z, \quad z \in G_k.$$

It is justified by means of the expansion

$$E_1(t-z) = \frac{1}{t-z} - \pi(t-z) - S_4(t-z)^3 + O(f^2). \tag{83}$$

The same arguments hold for an arbitrary iteration order p. More precisely, equation (3) can be written up to $O(f^2)$ in the form

$$\varphi_k^{(p)}(z) - \frac{\varrho}{2\pi i}\int_{L_k} \overline{\varphi_k^{(p)}(t)} E_1(t-z)\mathrm{d}t = z, \quad z \in G_k. \tag{84}$$

This equation coincides with (81) which has the unique solution for $|\varrho| < 1$ in the space $\mathcal{H}(G_k)$. It is also proved that equations (81) and (82) have solutions coinciding up to $O(f^2)$.

This proves the theorem.

Introduce the operator bounded in the space $\mathcal{H}(G_k)$ [23]

$$(\mathcal{S}_k h)(z) = \frac{1}{2\pi i}\int_{L_k} \frac{\overline{h(t)}}{t-z}\mathrm{d}t, \quad h \in \mathcal{H}(G_k). \tag{85}$$

Equation (82) has the unique solution in the space $\mathcal{H}(G_k)$ for $|\varrho| < 1$ [3]. It can be represented by the absolutely convergent in ϱ series [5, Chapter 2]

$$\varphi_k^{(1)}(z) = z + \sum_{\ell=1}^{\infty} \varrho^{\ell} \mathcal{S}_k^{\ell} z \equiv z + \varrho\phi_k^{(1)}(z), \quad z \in G_k \cup L_k, \tag{86}$$

where $\mathcal{S}_k^{\ell} z$ denotes the consequent application ℓ times of the operator $\mathcal{S}$ with the initial function $h(z) = z$.

Two components of the normalized effective permittivity tensor $\boldsymbol{\varepsilon}_{\perp}$ are calculated. Calculating the derivative $\frac{d}{dz}\varphi_k^{(1)}(z)$we obtain the formulas valid up to $O(f^2)$

$$\begin{aligned} \varepsilon_{11} &= 1 + 2\varrho f + 2\mathrm{Re}\ \textstyle\sum_{\ell=1}^{\infty} \varrho^{\ell+1} \sum_{k=1}^{N} \int_{G_k} \frac{dS_k^{\ell} z}{dz}(\xi_1 + i\xi_2)\ \mathrm{d}\xi_1 \mathrm{d}\xi_2, \\ \varepsilon_{12} &= -2\mathrm{Im}\ \textstyle\sum_{\ell=1}^{\infty} \varrho^{\ell+1} \sum_{k=1}^{N} \int_{G_k} \frac{dS_k^{\ell} z}{dz}(\xi_1 + i\xi_2)\ \mathrm{d}\xi_1 \mathrm{d}\xi_2. \end{aligned} \tag{87}$$

The solution of equation (82) is represented in terms of the series (86) in ϱ. An analogous representation holds for the solution of equation (81). Such a series yields the asymptotic formula (87) for effective constants up to $O(f^2)$. This confirm the precision up to $O(f^2)$ for $\varphi_k^{(1)}(z)$ chosen a priori at the beginning.

The integral equations (81) and (82) can be considered as a method to determine the complex potentials $\varphi_k^{(1)}(z)$ in inclusions. The complex potential $\varphi(z)$ in the host can be calculated. Analysis of this method yields the series (87) in ϱ, i.e., shows the structure of approximation and its precision. The formula (87) cannot be considered as a closed form expression until the infinite set of compositions of integral operators $\mathcal{S}_k^\ell z$ is not constructively found. One can apply another numerical method, FEM, to determine the local field in a composite with one inclusion G_k. But any method yields at most the term $1+2\varrho f$ plus the term of order $O(f)$ symbolically or numerically equivalent to the double sum $\sum_{\ell=1}^{\infty}\sum_{k=1}^{N}$ from (87) and does not concern the higher order term $O(f^2)$.

1.6. **Maxwell's self-consistent approach.** In the present Section, we outline Maxwell's self-consistent approach [8]

Consider one inclusion G_1 of permittivity ε_1 embedded in the host material of permittivity ε. It is assumed that the permittivity is real and the boundary $L_1 = \partial G_1$ is a piece-wise simple Lyapunov curve. The perfect contact between the components is expressed by equations

$$u(t) = u_1(t), \quad \frac{\partial u}{\partial \mathbf{n}}(t) = \varepsilon_1 \frac{\partial u_1}{\partial \mathbf{n}}(t), \quad t \in L_1, \tag{88}$$

where the functions $u_1(z)$ and $u(z)$ satisfy Laplace's equation in the domains G_1 and $G = \widehat{\mathbb{C}} \setminus (G_1 \cup L_1)$. Moreover, the functions are continuously differentiable in the closures of considered domains, and the function $u(z)$ has a singularity at infinity

$$u(z) \sim x_1, \quad \text{as } z \to \infty. \tag{89}$$

This singularity corresponds to the external potential written in the complex form

$$u_{ext}(z) = \operatorname{Re} z. \tag{90}$$

In the considered case of a two-phase composite, we have one dimensionless scalar contrast parameter

$$\varrho = \frac{\varepsilon_1 - 1}{\varepsilon_1 + 1}. \tag{91}$$

The considered problem is reduced to the scalar $\mathbb{R}$-linear problem

$$\varphi(t) = \varphi_1(t) - \varrho\overline{\varphi_1(t)}, \quad t \in L_1, \tag{92}$$

where the functions $\varphi(z)$ and $\varphi_1(z)$ are analytic in the domains G and G_1, respectively. Moreover, these functions are continuously differentiable in the closures of considered domains, and the function $\varphi(z)$ has a pole at infinity

$$\varphi(z) \sim z, \quad \text{as } z \to \infty. \tag{93}$$

Apply the Cauchy-type operator for L_1 to the boundary condition (92). We obtain the integral equation equivalent to (81) in the case $\theta_1 = 0$

$$\varphi_1(z) - \frac{\varrho}{2\pi i}\int_{L_1} \frac{\overline{\varphi_1(t)}}{t-z}\mathrm{d}t = z, \quad z \in G_1. \tag{94}$$

Application of the Cauchy-type operator (92) for $z \in G$ yields

(95) $$\varphi(z) = \frac{\varrho}{2\pi i} \int_{L_1} \frac{\overline{\varphi_1(t)}}{t-z} \mathrm{d}t + z, \quad z \in G.$$

Let $\varphi_1(z)$ be the solution to equation (94). Then, the complex potential $\varphi(z)$ is calculated by (95).

One can see that equations (81) and (94) are the same. Therefore, Maxwell's self-consistent approach leads to the same equation, obtained by applying the first-order iteration in the implicit scheme of Schwarz's method discussed in Section 1.5. It was established that the solution to equation (81) yields the formulas (87) valid up to $O(f^2)$. This justifies the validity of Maxwell's approach only in the first-order approximation in f. The claim often made in various works, stating that the first-order approximation could be extended to higher orders in f without accounting for higher-order terms $\varphi_1^{(p)}(z)$ $(p \geq 2)$, lacks any foundation.

Given its limitations, we continue to describe Maxwell's approach along the lines of [8]. Expand the complex potential $\varphi(z)$ near infinity in the Laurent series and take the coefficient on $-\frac{1}{z}$. The result value M_1 is called the dipole moment of the inclusion G_1 [18, 1, 21]. It follows from (95) that

(96) $$M_1 = \frac{\varrho}{2\pi \mathrm{i}} \int_{L_1} \overline{\varphi_1(t)} \mathrm{d}t.$$

Considering the single inclusion G_k in the complex plane by the above method, we introduce the dipole moments of G_k and estimate it using (86)

(97) $$M_k = \frac{\varrho}{2\pi \mathrm{i}} \int_{L_k} \overline{\varphi_k(t)} \mathrm{d}t = \varrho \frac{|G_k|}{\pi} + m_k', \quad k = 1, 2, \ldots, n,$$

where $m_k' = \frac{\varrho^2}{2\pi \mathrm{i}} \int_{L_k} \overline{\phi_k(t)} \mathrm{d}t$ is of order $O(\varrho^2 f)$. It is assumed that $n = n(R_0)$ inclusions lie in the disk $|z| < R_0$.

Let the disk $|z| < R_0$ be occupied by a material of the real permittivity ε_e and embedded in a medium with normalized unit permittivity. This problem is a particular case of the above problem (88)-(89) and can be written in the form

(98) $$U(t) = U_1(t), \quad \frac{\partial U}{\partial \mathbf{n}}(t) = \varepsilon_e \frac{\partial U_1}{\partial \mathbf{n}}(t), \quad |t| = R_0,$$

where the functions $U_1(z)$ and $U(z)$ satisfy Laplace's equation in the domains $|z| < R_0$ and $|z| > R_0$. The considered functions are continuously differentiable in the closures of considered domains, and the function $U(z)$ has the same singularity at infinity as $u(z)$

(99) $$U(z) \sim x_1, \quad \text{as } z \to \infty.$$

The problem (98)-(99) is reduced to the $\mathbb{R}$-linear problem

$$\Phi(t) = \Phi_1(t) - \varrho_e \overline{\Phi_1(t)}, \quad |t| = R_0, \tag{100}$$

where

$$\varrho_e = \frac{\varepsilon_e - 1}{\varepsilon_e + 1}. \tag{101}$$

The functions $\Phi(z)$ and $\Phi_1(z)$ are analytic in the domains $|z| < R_0$ and $|z| > R_0$, respectively. Moreover, these functions are continuously differentiable in the closures of considered domains, and the function $\Phi(z)$ has a pole at infinity

$$\Phi(z) \sim z, \quad \text{as } z \to \infty. \tag{102}$$

The problem (100)-(102) has the following solution up to an additive arbitrary constant

$$\Phi(z) = z - \frac{\varrho_e R_0^2}{z}, \quad \Phi_1(z) = z. \tag{103}$$

Suppose that n inclusions G_k ($k = 1, 2, \ldots, n$) are located inside the large disk $|z| < R_0$. Let the effective permittivity of the homogenized large disk $|z| < R_0$ be equal to the scalar ε_e. The dipole moment of $\Phi(z)$ equals $\varrho_e R_0^2$. Following Maxwell, we equate the sum of the dipole moments of G_k to the total dipole moment of the homogenized material of the large disk

$$\varrho_e R_0^2 = \sum_{k=1}^{n} M_k \equiv \varrho \sum_{k=1}^{n} \frac{|G_k|}{\pi} + nm', \tag{104}$$

where $m' = \frac{1}{n} \sum_{k=1}^{n} m'_k$. Taking into account (101) consider (104) as an equation on ε_e. Its solution yields the effective permittivity of the disk $|z| < R_0$

$$\varepsilon_e(n) = \frac{1 + R_0^{-2} \sum_{k=1}^{n} M_k}{1 - R_0^{-2} \sum_{k=1}^{n} M_k} = \frac{1 + \frac{\varrho}{\pi R_0^2} \sum_{k=1}^{n} |G_k| + nm' R_0^{-2}}{1 - \frac{\varrho}{\pi R_0^2} \sum_{k=1}^{n} |G_k| + nm' R_0^{-2}}. \tag{105}$$

It is assumed that $\varepsilon_e(n)$ tends to the effective permittivity of composite ε_e, as $n \to \infty \Leftrightarrow R_0 \to \infty$. In particular, it is assumed that the concentration is properly defined

$$f = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{n|G_k|}{\pi R_0^2}. \tag{106}$$

Bear in mind that the homogenized medium is initially assumed macroscopically isotropic.

Remark 2. *The existence of the concentration, the limit* (106)*, does not guarantee that a composite is homogenized, i.e., there exists the limit* $\varepsilon_e = \lim_{n\to\infty} \varepsilon_e(n)$ *even in the second-order approximation* $O(f^2)$ [10]*. This implies that any universal formula for* ε_e *holds only up to* $O(f^2)$ *except at neutral inclusions. The devil is in the details, in the term of order* f^2*.*

This mathematical curiosity arises in self-consistency. The concept becomes clear through the analysis of two half-planes $x_1 > 0$ and $x_1 < 0$ comprised of square and hexagonal arrays of disks having the same concentration. These arrays maintain the same concentration but have different effective permittivity.

Consider the case when $|G_1| = |G_2| = \ldots = |G_N|$ and $m'_1 = m'_2 = \ldots = m'_N$. Then, (105) yields

$$\varepsilon_e = \frac{1 + \varrho f + m' f \pi |G_1|^{-1}}{1 - \varrho f - m' f \pi |G_1|^{-1}} + O(f^2). \tag{107}$$

One can check that the term $m' f \pi |G_1|^{-1}$ is of order $O(\varrho^2 f)$.

Let equal disks of radius r be embedded in the host materials. Using the relation $t = \frac{r^2}{\bar{t}}$ on the circle $|t| = r$ rewrite equation (94)

$$\varphi_1(z) - \frac{\varrho}{2\pi i} \int_{|t|=r} \frac{\overline{\varphi_1\left(\frac{r^2}{\bar{t}}\right)}}{t - z} \mathrm{d}t = z, \quad |z| < r. \tag{108}$$

By the symmetry principle the function $\overline{\varphi_1\left(\frac{r^2}{\bar{t}}\right)}$ is analytically continued into the domain $|z| > r$. We calculate the integral from (108) and obtain

$$\varphi_1(z) = \varrho \overline{\varphi_1(0)} + z, \quad |z| < r. \tag{109}$$

Determine $\varphi(z)$ by (92) up to an additive constant

$$\varphi(z) = z - \frac{\varrho r^2}{z}, \quad |z| > r. \tag{110}$$

Therefore, the dipole for the disk $M_1 = \varrho r^2$ and $m'_1 = 0$ by (97). Then, (107) becomes the famous Maxwell-Garnett (Clausius-Mossotti, Maxwell, Lorenz, Lorentz, Maxwell-Garnett) approximation

$$\varepsilon_e = \frac{1 + f\varrho}{1 - f\varrho} + O(f^3). \tag{111}$$

The precision is increased here according to [13] established for equal disks.

Some comments are needed to explain the formula (111). For any positive f, it is easy to present such a location of perfectly conducting disks ($\varrho = 1$) that a collection C_M will contain two percolation chains along the axes. Hence, the averaged permittivity of C_M will be infinite. It cannot be reached by the approximation (111). Conversely, any macroscopically isotropic composite with circular inclusions satisfies (111) [13]. This assertion does not contradict the previous observation since the formula (111) is asymptotic with a positive infinitesimally small number f. In this sense, the asymptotic formula (111) is universal and holds for sufficiently small f.

Let the ellipse L_1 be defined by equation $\frac{x_1^2}{(1+\alpha)^2}+\frac{x_2^2}{(1-\alpha)^2}=1$, where for definiteness $0<\alpha<1$. The semi-axes of the ellipse are equal to $(1+\alpha)$ and $(1-\alpha)$. The ellipse equation can be written in the complex form

$$\overline{z}=z\frac{1+\alpha^2}{2\alpha}-\frac{1-\alpha^2}{2\alpha}\sqrt{z^2-4\alpha},\quad z\in L_1, \tag{112}$$

where the square root is defined on the complex plane without the slit $(-2\sqrt{\alpha},2\sqrt{\alpha})$ on the x_1-axis. Moreover, the square root tends to $+\infty$ as $z=x_1>0$ tends to infinity along the real axis.

Consider the $\mathbb{R}$-linear problem (92)-(93). One can check that the functions

$$\varphi_1(z)=\frac{z}{1-\varrho\alpha},\ \varphi(z)=\frac{1}{1-\varrho\alpha}\left[z\left(1-\varrho\frac{1+\alpha^2}{2\alpha}\right)+\varrho\frac{1-\alpha^2}{2\alpha}\sqrt{z^2-4\alpha}\right] \tag{113}$$

satisfy the problem (92)-(93). The dipole moment of the ellipse has the form

$$M_1=\varrho\frac{1-\alpha^2}{1-\varrho\alpha}. \tag{114}$$

Consider equal non-overlapping elliptic inclusions G_k obtained from G_1 by translation and rotation by an angle θ_k. The complex potentials for the inclusion G_k can be obtained from (113) by the transformation $z\mapsto z\exp(\mathrm{i}\theta_k)+a_k$. The dipole moment has the form

$$M_k=\varrho\frac{1-\alpha^2}{1-\varrho^2\alpha^2}\left[1+\exp(\mathrm{i}\theta_k)\varrho\alpha\right]. \tag{115}$$

Let inclination angle θ be considered a random variable uniformly distributed on $(0,\pi)$. Then, the mathematical expectation $\mathbf{E}[\theta]=0$ and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}M_k=\frac{\varrho|G_1|}{\pi}\frac{1}{1-\varrho^2\alpha^2}, \tag{116}$$

since the area of ellipse holds $|G_1|=\pi(1-\alpha^2)$. Using this formula, we obtain from (105) and (106)

$$\varepsilon_e=\frac{1+\frac{\varrho f}{1-\varrho^2\alpha^2}}{1-\frac{\varrho f}{1-\varrho^2\alpha^2}}+O(f^2)=1+\frac{2\varrho f}{1-\varrho^2\alpha^2}+O(f^2). \tag{117}$$

2. Effective permittivity tensor for multi-phase composites

Above in this note, we derived formulas for real-valued permittivity. We now begin to explore complex values of the normalized permittivity of components ε_k. Consequently, the contrast parameter ϱ_k also becomes complex. Building upon the foundations laid out, we extend these formulas to the complex domain through analytic continuation in terms of ϱ_k. To ensure the validity of this continuation, it is imperative that we establish a groundwork for understanding the interplay between complex permittivity and its matrix representation. This will be our main goal as we move forward.

The effective permittivity tensor with complex components will be denoted in the same way

$$\boldsymbol{\varepsilon}_{\perp} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix}, \tag{118}$$

where $\varepsilon_{12} = \varepsilon_{21}$.

Introduce the averaged value of a function $F(z)$ over the unit periodicity cell Q

$$\langle F(z) \rangle = \int_Q F(x_1 + ix_2)\, \mathrm{d}x_1 \mathrm{d}x_2. \tag{119}$$

Here, the real plane coordinates $\mathbf{x} = (x_1, x_2)$ are identified with the complex coordinate $z = x_1 + \mathrm{i}x_2$. The complex effective permittivity tensor can be defined through the following relation

$$\langle \varepsilon(z) \nabla u(z) \rangle = \boldsymbol{\varepsilon}_{\perp} \langle \nabla u(z) \rangle, \tag{120}$$

where $\varepsilon(z) = 1$ in G and $\varepsilon(z) = \varepsilon_k$ in G_k. The value $\nabla u_{ext} := \langle \nabla u(z) \rangle$ determines the complex external flux applied to the unit periodicity cell Q. It follows from the theory of homogenization [2, 6] that in order to determine the tensor $\boldsymbol{\varepsilon}_{\perp}$, it is sufficient to find two local fields $\nabla u(z)$ for two external fluxes

$$\nabla u_{ext[1]} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \nabla u_{ext[2]} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{121}$$

Let us fix the first external flux from (121). Consider the vector equality with complex components

$$\langle \nabla u \rangle = \int_G \nabla u \, \mathrm{d}x_1 \mathrm{d}x_2 + \sum_{k=1}^{N} \int_{G_k} \nabla u_k \mathrm{d}x_1 \mathrm{d}x_2. \tag{122}$$

Transform the double integrals by the divergence theorem associated to the Ostrogradsky-Gauss theorem of real vector analysis

$$\int_G \nabla u \, \mathrm{d}x_1 \mathrm{d}x_2 = \int_{\partial Q} u\mathbf{n} \, \mathrm{d}s + \sum_{k=1}^{N} \int_{\partial G_k} (u_k - u)\mathbf{n} \, \mathrm{d}s, \tag{123}$$

where $\mathbf{n}$ denotes the outward normal unit vector to the boundary $\partial G = \partial Q - \sum_{k=1}^{N} \partial G_k$. It follows from the quasi-periodicity conditions that the integrals over ∂G_k vanish. The integral over ∂Q is calculated by the jumps of u per unit periodicity cell Q, see for details Chapter 3 and Remark 5 in Section 2.2.2 of [5]. Ultimately, we have

$$\langle \nabla u(z) \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \boldsymbol{\varepsilon}_{\perp} \langle \nabla u(z) \rangle = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{21} \end{pmatrix}. \tag{124}$$

We now proceed to calculate the left part of (120) which has to be equal to the second vector of (124)

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{21} \end{pmatrix} = \int_G \nabla u \; \mathrm{d}x_1 \mathrm{d}x_2 + \sum_{k=1}^N \varepsilon_k \int_{G_k} \nabla u_k \mathrm{d}x_1 \mathrm{d}x_2. \tag{125}$$

Greens formula

$$\int_S \left(\frac{\partial F}{\partial x_1} - \frac{\partial G}{\partial x_2} \right) \mathrm{d}x_1 \mathrm{d}x_2 = \int_{\partial S} G \; \mathrm{d}x_1 + F \; \mathrm{d}x_2. \tag{126}$$

The components of (125) can be written in the form

$$\varepsilon_{11} = \int_{\partial Q} u \; \mathrm{d}x_2 + \sum_{k=1}^N (\varepsilon_k - 1) \int_{\partial G_k} u_k \; \mathrm{d}x_2, \tag{127}$$

$$-\varepsilon_{21} = \int_{\partial Q} u \; \mathrm{d}x_1 + \sum_{k=1}^N (\varepsilon_k - 1) \int_{\partial G_k} u_k \; \mathrm{d}x_1. \tag{128}$$

Using the jump conditions per unit periodicity cell Q we calculate the integrals over ∂Q

$$\int_{\partial Q} u \; \mathrm{d}x_1 = - \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[u\left(x_1 + \frac{\mathrm{i}}{2} \right) - u\left(x_1 - \frac{\mathrm{i}}{2} \right) \right] \mathrm{d}x_1 = 0, \tag{129}$$

$$\int_{\partial Q} u \; \mathrm{d}x_2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[u\left(\frac{1}{2} + \mathrm{i}x_2 \right) - u\left(-\frac{1}{2} + \mathrm{i}x_2 \right) \right] \mathrm{d}x_2 = 1. \tag{130}$$

Again, applying Greens formula for G_k we obtain

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{k=1}^N (\varepsilon_k - 1) \int_{G_k} \nabla u_k \; \mathrm{d}x_1 \mathrm{d}x_2. \tag{131}$$

We have the representation

$$\psi_k(z) = \frac{1}{2}(I + \boldsymbol{\alpha}_k) \begin{pmatrix} \frac{\partial u_k'}{\partial x_1} - \mathrm{i}\frac{\partial u_k'}{\partial x_2} \\ \\ \frac{\partial u_k''}{\partial x_1} - \mathrm{i}\frac{\partial u_k''}{\partial x_2} \end{pmatrix}, \quad z \in G_k, \tag{132}$$

which can be written in the equivalent form

$$\begin{pmatrix} \frac{\partial u_k'}{\partial x_1} - \mathrm{i}\frac{\partial u_k'}{\partial x_2} \\ \\ \frac{\partial u_k''}{\partial x_1} - \mathrm{i}\frac{\partial u_k''}{\partial x_2} \end{pmatrix} = \frac{2}{|1 + \varepsilon_k|^2} \begin{pmatrix} 1 + \varepsilon_k' & \varepsilon_k'' \\ -\varepsilon_k'' & 1 + \varepsilon_k' \end{pmatrix} \psi_k(z), \quad z \in G_k. \tag{133}$$

Then, the gradient ∇u_k can be written in terms of the complex potentials

$$\nabla u_k \equiv \begin{pmatrix} \frac{\partial u_k'}{\partial x_1} + \mathrm{i}\frac{\partial u_k''}{\partial x_1} \\ \\ \frac{\partial u_k'}{\partial x_2} + \mathrm{i}\frac{\partial u_k''}{\partial x_2} \end{pmatrix} = \frac{2}{\varepsilon_k + 1} \begin{pmatrix} \mathrm{Re}\,\psi_{1k} + \mathrm{iRe}\,\psi_{2k} \\ \\ -\mathrm{Im}\,\psi_{1k} - \mathrm{iIm}\,\psi_{2k} \end{pmatrix}. \tag{134}$$

where $\psi_{1k}(z)$ and $\psi_{2k}(z)$ are coordinate of the vector function $\psi_k(z)$.

Substitute (134) into (131)

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\sum_{k=1}^{N} \varrho_k \int_{G_k} \begin{pmatrix} \mathrm{Re}\,\psi_{1k} + \mathrm{iRe}\,\psi_{2k} \\ -\mathrm{Im}\,\psi_{1k} - \mathrm{iIm}\,\psi_{2k} \end{pmatrix} \mathrm{d}x_1 \mathrm{d}x_2, \tag{135}$$

where the scalar values ϱ_k can be found [12]. Therefore, in order to calculate (135) we have to find the vector function $\frac{d}{dz}\varphi_k(z) = \psi_k(z) = (\psi_{1k}(z), \psi_{2k}(z))^\top$ satisfying the considered boundary value problem.

3. Integral equations and their approximate solution for multi-phase composites

The scalar $\mathbb{R}$-linear problem was reduced to the system of scalar integral equations. Using the same arguments, we reduce the vector $\mathbb{R}$-linear problem to the vector system of integral equations up to an arbitrary additive constant vector

$$\boldsymbol{\varphi}_k(z) = \sum_{m=1}^{N} \frac{1}{2\pi i} \int_{L_m} \boldsymbol{\beta}_m \overline{\boldsymbol{\varphi}_m(t)} E_1(t-z)\, \mathrm{d}t + \begin{pmatrix} z \\ 0 \end{pmatrix}, \; z \in G_k \; (k = 1, 2, \ldots, N). \tag{136}$$

In the scalar case, the contrast approximations were defined by the constant $\varrho_0 = \max_{k=1,2,\ldots,N} |\varrho_k|$. In the vector-matrix case, introduce the spectral norm

$$\varrho_0 = \max_{m=1,2,\ldots,N} \left| \frac{\varepsilon_m - 1}{\varepsilon_m + 1} \right|. \tag{137}$$

Following Section 1.3, we find an approximate solution to equations (136) by two iterations. Similar to (29) the first-order term has the form

$$\boldsymbol{\varphi}_k^{(1)}(z) = \begin{pmatrix} z \\ 0 \end{pmatrix} + O(1), \quad z \in G_k. \tag{138}$$

The second order approximation is found similar to (31)

$$\boldsymbol{\varphi}_k^{(2)}(z) = \sum_{m=1}^{N} \frac{\boldsymbol{\beta}_m}{2\pi i} \int_{L_m} \begin{pmatrix} \overline{t} \\ 0 \end{pmatrix} E_1(t-z)\, \mathrm{d}t + \begin{pmatrix} z \\ 0 \end{pmatrix} + O(\varrho_0^2), \quad z \in G_k. \tag{139}$$

Below, we omit the precision in ϱ_0 for shortness and keep track of the precision in f. Differentiate equation (139)

(140) $$\psi_k^{(2)}(z) = \frac{d}{dz}\boldsymbol{\varphi}_k^{(2)}(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{m=1}^{N} \frac{\boldsymbol{\beta}_m}{2\pi i} \int_{L_m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bar{t}\, E_2(t-z)\, \mathrm{d}t,$$

Using the relation

(141) $$\boldsymbol{\beta}_m \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{|1+\varepsilon_m|^2} \begin{pmatrix} |\varepsilon_m|^2 - 1 \\ 2\varepsilon_m'' \end{pmatrix}$$

write (140) in the form

(142) $$\psi_k^{(2)}(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\pi} \sum_{m=1}^{N} \frac{1}{|1+\varepsilon_m|^2} \begin{pmatrix} |\varepsilon_m|^2 - 1 \\ 2\varepsilon_m'' \end{pmatrix} \frac{dg_m}{dz}(z),\ z \in G_k.$$

where $g_m(z)$ has the form (33), hence,

(143) $$\frac{dg_m}{dz}(z) = \frac{1}{2i} \int_{L_m} \bar{t}\, E_2(t-z)\, \mathrm{d}t.$$

Write the relation (142) by coordinates

(144) $$\begin{aligned} \psi_{1k}^{(2)}(z) &= 1 + \tfrac{1}{\pi} \textstyle\sum_{m=1}^{N} \frac{|\varepsilon_m|^2-1}{|1+\varepsilon_m|^2} \frac{dg_m}{dz}(z) \\ \psi_{2k}^{(2)}(z) &= \tfrac{1}{\pi} \textstyle\sum_{m=1}^{N} \frac{2\varepsilon_m''}{|1+\varepsilon_m|^2} \frac{dg_m}{dz}(z), \quad z \in G_k. \end{aligned}$$

Taking into account the identity

(145) $$\varrho_m \equiv \frac{\varepsilon_m - 1}{\varepsilon_m + 1} = \frac{|\varepsilon_m|^2 - 1 + 2\mathrm{i}\varepsilon_m''}{|1+\varepsilon_m|^2},$$

find the vector

(146) $$\begin{pmatrix} \mathrm{Re}\,\psi_{1k}^{(2)} + \mathrm{iRe}\,\psi_{2k}^{(2)} \\ -\mathrm{Im}\,\psi_{1k}^{(2)} - \mathrm{iIm}\,\psi_{2k}^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\pi} \sum_{m=1}^{N} \varrho_m \begin{pmatrix} \mathrm{Re}\, \frac{dg_m}{dz} \\ -\mathrm{Im}\, \frac{dg_m}{dz} \end{pmatrix}.$$

Introduce the averaged scalar contrast parameter over inclusions similar to the case of real-valued permittivity (35)

(147) $$\langle \varrho \rangle := \sum_{k=1}^{N} \varrho_k |G_k|.$$

Substitute (146) into (135)

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{21} \end{pmatrix} = (1+2\langle\varrho\rangle)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{148}$$

$$+\frac{2}{\pi}\sum_{k=1}^{N}\sum_{m=1}^{N}\varrho_k\varrho_m\int_{G_k}\begin{pmatrix} \text{Re } h_{km} \\ -\text{Im } h_{km} \end{pmatrix} \mathrm{d}x_1\mathrm{d}x_2 + O(\varrho_0^3 f^{7/2}),$$

where h_{km} has the form (37) and ϱ_0 is given by (137). The precision of (148) is taken as the precision of (36) due to the same application of Schwarz's scheme. One can see that the same values h_{km} are needed for the calculation of the permittivity tensor in the case of real-valued permittivity by (36) and of complex permittivity by (148). The formulas (36) and (148) are similar, and (148) has to include (36) as a particular case. The difference between (36) and (148) consists in the assumption on the contrast parameters ϱ_k which have to be real in (36) and can be complex in (148).

This observation allows us to state the following assertion

Rule $\mathbb{R} \to \mathbb{C}$: Any formula for the real-valued permittivity of components is transformed into a formula for the complex permittivity.

This Rule is loosely stated and needs explanations since a formula for the real-valued permittivity must first be written in the corresponding form to be extended to the complex permittivity. The formulas from [5, 14] and others were obtained using the classic scalar complex potentials. For instance, the formula (36) for the real-valued permittivity was written in the form

$$\varepsilon_{11} - \mathrm{i}\varepsilon_{21} = 1 + 2\langle\varrho\rangle + \frac{2}{\pi}\sum_{k=1}^{N}\sum_{m=1}^{N}\varrho_k\varrho_m\int_{G_k} h_{km}\,\mathrm{d}x_1\mathrm{d}x_2 + O(\varrho_0^3 f^{7/2}). \tag{149}$$

Hence, two real equations (36) are written in the form of one complex equation. In order to extend (149) to the complex permittivity, first, one has to write (149) in the vector form (148). After this, one can extend the real-valued vector to the complex-valued vector, assuming that the contrast parameters ϱ_k become complex.

Now, we take the second external flux (121). Analogously the previous transformations, we arrive at the formula

$$\begin{pmatrix} \varepsilon_{22} \\ -\varepsilon_{12} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\sum_{k=1}^{N}\varrho_k\int_{G_k}\begin{pmatrix} \text{Re } \psi_{1k} + i\text{Re } \psi_{2k} \\ -\text{Im } \psi_{1k} - i\text{Im } \psi_{2k} \end{pmatrix} \mathrm{d}x_1\mathrm{d}x_2, \tag{150}$$

where ψ_{1k} and ψ_{2k} differ from the values of (135) calculated for the first flux. Two vector equations (135) and (148) can be written in the form

$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2\sum_{k=1}^{N} \varrho_k \times$$

(151)

$$\int_{G_k} \begin{pmatrix} \mathrm{Re}\,\psi_{1k[1]} + \mathrm{iRe}\,\psi_{2k[1]} & \mathrm{Im}\,\psi_{1k[2]} + \mathrm{iIm}\,\psi_{2k[2]} \\ -\mathrm{Im}\,\psi_{1k[1]} - \mathrm{iIm}\,\psi_{2k[1]} & \mathrm{Re}\,\psi_{1k[2]} + \mathrm{iRe}\,\psi_{2k[2]} \end{pmatrix} \mathrm{d}x_1 \mathrm{d}x_2,$$

where the subscripts [1] and [2] indicate the solutions constructed by the corresponding external fields (121).

The formula (70) was derived for the real-valued permittivity. Using **Rule** $\mathbb{R} \to \mathbb{C}$ one can apply this formula to the complex permittivity

$$\boldsymbol{\varepsilon}_\perp = (1 + 2\langle\varrho\rangle)I + 2f \begin{pmatrix} \mathrm{Re}\,\mathcal{J}_0 & -\mathrm{Im}\,\mathcal{J}_0 \\ -\mathrm{Im}\,\mathcal{J}_0 & -\mathrm{Re}\,\mathcal{J}_0 \end{pmatrix}$$

(152)

$$+2f^2 \begin{pmatrix} \mathrm{Re}\,\mathcal{L}_0 & -\mathrm{Im}\,\mathcal{L}_0 \\ -\mathrm{Im}\,\mathcal{L}_0 & 2 - \mathrm{Re}\,\mathcal{L}_0 \end{pmatrix} + 2f^{5/2} \begin{pmatrix} \mathrm{Re}\,\mathcal{V}_{01} & -\mathrm{Im}\,\mathcal{V}_{01} \\ -\mathrm{Im}\,\mathcal{V}_{01} & -\mathrm{Re}\,\mathcal{V}_{01} \end{pmatrix}$$

$$+2f^3 \begin{pmatrix} \mathrm{Re}\,\mathcal{V}_{02} & -\mathrm{Im}\,\mathcal{V}_{02} \\ -\mathrm{Im}\,\mathcal{V}_{02} & -\mathrm{Re}\,\mathcal{V}_{02} \end{pmatrix} + 2f^{7/2} \begin{pmatrix} \mathrm{Re}\,\mathcal{V}_{03} & -\mathrm{Im}\,\mathcal{V}_{03} \\ -\mathrm{Im}\,\mathcal{V}_{03} & -\mathrm{Re}\,\mathcal{V}_{03} \end{pmatrix}$$
$$+O(\varrho_0^3 f^4),$$

where the values from (152) are calculated by the same formulas (63) and (56)-(61).

The first invariant of the tensor $\boldsymbol{\varepsilon}_\perp$ takes the form

(153) $$\frac{1}{2}(\varepsilon_{11} + \varepsilon_{22}) = 1 + 2\langle\varrho\rangle + 2\langle\varrho\rangle^2 + O(\varrho_0^3).$$

One can see that it does not depend on the location of inclusions in the considered approximation.

The normalized longitudinal permittivity of the considered fibrous composite is calculated by the mean value, see formula (3.142) from [25]

(154) $$\varepsilon_\| = 1 - f + \sum_{k=1}^{N} \varepsilon_k f_k,$$

where f_k stands for the concentration of the phase of permittivity ε_k.

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