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MATHEMATICAL MODELLING OF HIGHLY DISORDERED ANISOTROPIC STRUCTURES.
PART 2.1. NEW THEORIES ENABLING HOMOGENIZATION OF DISODERED METAMATERIALS.

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Abstract

We develop a new computationally effective method based on the structural sums (generalized Eisenstein-Rayleigh sums) in the framework of homogenization theory. The method can be applied to various multi-phase dispersed metamaterials. It can be considered as a fast computational method alternative to high-order spatial correlation functions. The method will be applied to detect a difference in statistical features between normal (health) and anomalous (ill) types of cells. The method is considered as the Schwarz alternating algorithm. Its various implementations are applied to determine the local fields in 2D dispersed composites. Following the homogenization theory, we consider a doubly periodic representative cell Q with an arbitrary number of inclusions per cell. The method of complex potentials and constructive results on the $\mathbb{R}$-linear problem are systematically applied.

1. On Schwarz's Method in the constructive homogenization theory

The theory of composites includes a broad spectrum of applied mathematics, in particular, asymptotic homogenization theory and both analytical and numerical computation of effective material properties. Homogenization theory leads to the statement of the periodic boundary value problem. It provides a rationale for employing local fields derived from the boundary value problem to determine the effective constants. The periodic statement yields the proper definition of effective constants through the integrals over the periodicity cell from the local fields [7, 35]. A composite with one inclusion per unit periodicity cell Q is called a regular periodic composite. A composite with $N > 1$ inclusions per unit periodicity cell is called a periodic composite. In the homogenization theory of random composites [35, 27, 75] based on the theory of measure, the periodicity is replaced by the spatial stationary process when the statistical properties of the medium are invariant under translations. In this case, a unit periodicity cell Q exists, and it represents the considered random composite [26, Chapter 3]. This is the postulate of the homogenization theory of random composites. Therefore, the proper way to determine effective constants of a random composite is to consider a periodized part of the structure [71, 72] with increasing N, keeping the distribution of location up to the stabilization of the computed constants. In this part, we show that the violation of periodization can lead to methodologically misleading results.

The homogenization theory is associated with the concepts of existence and uniqueness of effective constants. The estimation of the effective constants is the next problem of homogenization. In this part, we analyze approximate and exact analytical formulas. We call this area the constructive theory of composites, emphasizing the derivation of formulas for effective constants. Leave aside now other ways to achieve the same goal, namely, the theory of bounds [13, 46], pure numerical simulations, and the theory of Representative Volume Element (RVE).

Some mathematical theories and algorithms were first proposed due to physical intuition. The way from a physical idea to a rigorous mathematical method may be extended and contradictory. In this part, we discuss the reasonable approximations for effective permittivity (conductivity, permeability, elastic constants, etc.) Often, new formulas derived using intuition are reduced to the classical lower-order approximations with redundant higher-order tails; some formulas are correct, and some are wrong. The absence of precision analysis and numerical investigations of conditionally convergent series explains the main theoretical discrepancy.

In the present note, we determine the complex effective permittivity tensor for 2D (two-dimensional) composites [55]

(1) $$\boldsymbol{\varepsilon}_\perp = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix}.$$

In the case of 3D fibrous composites, the effective permittivity tensor is decomposed onto the transversal part perpendicular to fibers $\boldsymbol{\varepsilon}_\perp$ and the longitudinal permittivity $\boldsymbol{\varepsilon}_{||}$ parallel to fibers.

One of the seemingly natural and straightforward ways to compute $\boldsymbol{\varepsilon}_\perp$ is based on the determination of the local fields around a finite collection of M inclusions in the space with the further suggestion that M tends to infinity. It is not equivalent to the limit $N \to \infty$ in the extending periodicity cell since the obtained series converges conditionally. Therefore, the order of its summation can lead to any result. This is the crucial reason for diverse models to determine $\boldsymbol{\varepsilon}_\perp$ and announce a new model (formula) obtained as a partial sum of absolutely divergent series. First, this problem of the conditionally convergent series was resolved by Rayleigh [68] in 1892. Rayleigh did not present any explanation and just indicated the proper method of summation (Eisenstein's approach proposed in 1848 [78]). Some historical notes can be found in [53].

We come back to the permittivity (1) in Section 3 after introduction to Schwarz's method. In 1869-1870, Hermann Schwarz proposed the alternating method for solving the Dirichlet problem for the union of two overlapping domains. S.G. Mikhlin [45] extended the classical alternating method to boundary value problems for a multiply connected domain with holes/inclusions. The method is based on the decomposition of the considered domain with complex geometry onto simply connected domains and subsequent estimations of the field in a simply connected domain induced by fields in other domains. Though the main idea by Schwarz presents in Mikhlin's approach, the case of non-overlapping inclusions essentially differs from the original method for overlapping domains and decomposition methods [74, 25, 32]. The generalized alternating method of Schwarz can be presented as an infinite sequence of all the mutual interactions between the inclusions in the considered composite. In this book, we use the term Schwarz's method for shortness but only mean non-overlapping inclusions considered by Mikhlin.

In 1949, Mikhlin [45] proved the convergence of Schwarz's method for a doubly connected domain. Schwarz's method was independently applied to two circular inclusions in many works without referencing the original method. For example, it is called the iterating analytic self-maps of disks in [11]. The authors of [34] called Schwarz's method and its implementation by "heterogenization technique." The complete closed-form[1] solution to the problem for two circular holes was derived in [70].

Schwarz's method was slightly modified in [60, 61] so that its convergence was established for any multiply connected domain for Laplace's and other equations. The sequential steps of Schwarz's method can be presented in symbolic form. This leads to analytical approximate formulas for the local fields and for the effective constants of composites. The interactions between circular (spherical) inclusions were written by Kelvin's reflections on circles, and the effective constants were obtained in terms of Eisenstein functions [61, 26, 21].

2. General Schwarz's scheme

Integral equations in a Banach space corresponding to Schwarz's method were derived in [45] and modified in [60, 61]. Below, we present these equations in the general operator form on the potentials u_k in G_k, omitting some mathematical details described in the next sections. Let u_0 denote the given external potential. For simplicity, consider a two-phase dispersed composite with non-overlapping inclusions G_k of permittivity ε_1 embedded in the host of permittivity ε. In this note, up to Section 4, we will delve into Schwarz's method, presenting its basic idea, taking

[1] Unfortunately, the terms "closed-form" and "exact" solutions have been incongruously applied in some works. So, we are forced to stress that [70] contains a closed-form solution in the proper sense [4].

for simplicity real constants ε_1 and ε. Introduce the dimensionless contrast parameter

$$\varrho = \frac{\varepsilon_1 - \varepsilon}{\varepsilon_1 + \varepsilon}, \tag{2}$$

which $|\varrho| \le 1$ for ε_1 and ε changing from zero to $+\infty$.

The potential u_k is related to others u_m by the linear operator equation [58]

$$u_k = \varrho F_k u_k + \varrho \sum_{m \neq k} F_m u_m + u_0, \quad \text{in } G_k, \quad k = 1, 2, \ldots, N, \tag{3}$$

where $F_m u_m$ denotes the field in the domain G_k induced by the inclusion G_m. The term $F_k u_k$ produces the self-induced field. Equation (3) will be written explicitly in Section 5. The physical contrast parameter ϱ is a multiplier of the bounded operators F_m. The operators F_m are determined by the microstructure of the considered composite, which implicitly depends on the concentration of inclusions f. After undergoing a constructive homogenization procedure, the explicit dependence on the variable f can be represented.

Various explicit and implicit iterative schemes [25], including the splitting of the operators F_m, can be applied to equation (3). For instance, the self-induced field can be included in determining the field u_k. This scheme can be expressed in terms of the inverse operator

$$u_k = \varrho (I - \varrho F_k)^{-1} \left[\sum_{m \neq k} F_m u_m + u_0 \right] \quad \text{in } G_k, \quad k = 1, 2, \ldots, N, \tag{4}$$

where I stands for the identity operator. The construction of the inverse operator is reduced to a solution to the problem for one inclusion G_k. In the present note, we develop an asymptotic method in order to find the inverse operator in a constructive form.

The method of successive approximations for equation (3) leads to the contrast expansion

$$\begin{aligned} &u_k = u_0 + \varrho \textstyle\sum_{k_1} F_{k_1} u_0 + \varrho^2 \sum_{k_1,k_2} F_{k_1} F_{k_2} u_0 + \varrho^3 \sum_{k_1,k_2} F_{k_1} F_{k_2} F_{k_3} u_0 + \ldots \\ &\text{in } G_k \cup L_k, \end{aligned} \tag{5}$$

where $k, k_s = 1, 2, \ldots, N$ for $s = 1, 2, \ldots$.

After determining the local field, the effective constants can be found following the homogenization theory [7]. In the case of dispersed composites, the effective constants can be calculated by the averaged local field in inclusions [26, Chapter 3]. The local field can be found using various numerical and analytical methods, for instance, by FEM [20, 39], by integral equations [28, 45], by asymptotic and perturbation methods [5, 6, 19]. The effective properties can be estimated by the Hashin-Shtrikman bounds and their generalizations [46, 13]. Contrast and cluster expansions [46, 76, 26, 21] are applied to derive the macroscopic properties of composites.

We concentrate our attention on the asymptotic analysis of Schwarz's method for dispersed composites. This study yields a constructive method to derive analytical formulas for the effective

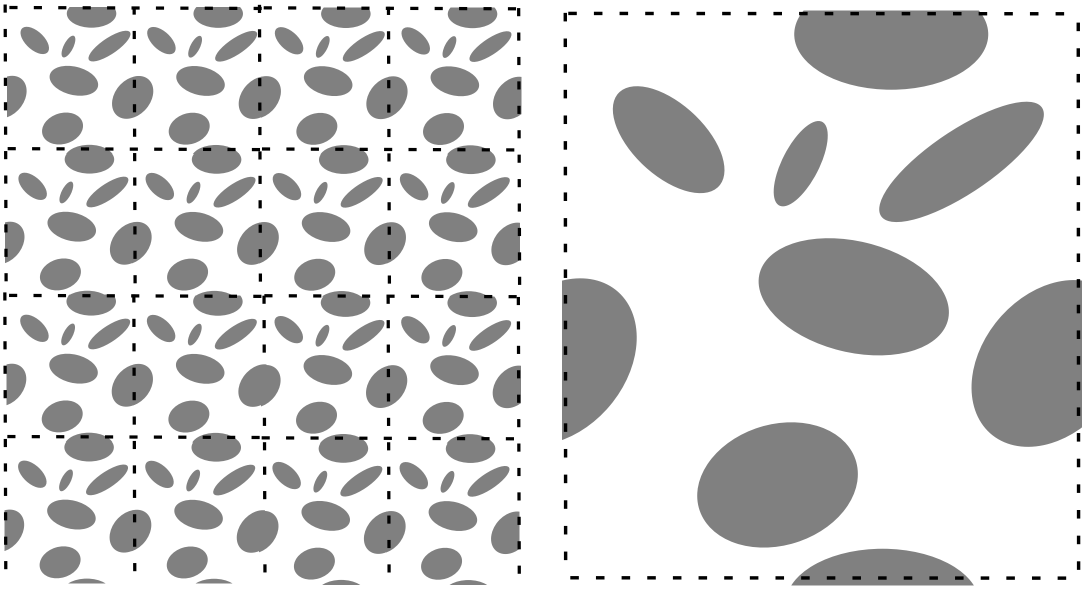

FIGURE 1. Doubly periodic composite and its representative square cell Q (RVE).

constants with the exactly calculated precision orders [58, 51]. The 2D permittivity problem is equivalent to conductivity and an anti-plane elastic problem. The elasticity problem requires two contrast parameters [54] that complicate computations. The same scheme outlined by equations (3)-(4) can be applied to 3D composites, including thermoelastic, piezoelectric, and other physical phenomena [69, 26, 21, 3].

Schwarz's method can be outlined as follows. First, we order the steps of the method by the sequence $\{\{0\},\{k_1\},\{k_1,k_2\},\ldots,\{k_1,k_2,\ldots,k_s\},\ldots\}$, where k_s runs over the numbers of inclusions $1,2,\ldots,N$ and $s=0,1,\ldots$. The number s is called the level of step. The zero level potential $U_k^{(0)}$ in G_k is equal to the external potential $u_0=x_1$ for which $\mathbf{E}_0=(-1,0)$ is calculated as the gradient $\mathbf{E}_0=-\nabla u_0$. Let us fix an inclusion G_{k_1} and introduce the potential U_{k_1} induced by $U_m^{(0)}$ $m=1,2\ldots,N$. The induced terms $F_m U_m^{(0)}$ correspond to the first iteration in equations (3). This first step has the level $s=1$ and includes N elements U_{k_1} $(k_1=1,2\ldots,N)$. The second step of level $s=2$ includes N^2 elements U_{k_1,k_2} $(k_1,k_2=1,2\ldots,N)$. The element U_{k_1,k_2} in the inclusion G_{k_1} goes back to the impact of the inclusion G_{k_2} onto G_{k_1}, i.e., first, the impact of the zero potential $U_{k_1}^{(0)}$ in G_{k_1} to the potential U_{k_2} in G_{k_2} and next the impact of U_{k_2} to the field in G_{k_1}. It is worth noting that the elements U_{k_1,k_2} and U_{k_2,k_1} do not coincide since they are defined in different inclusions. For example, the genesis of $U_{1,3,2,3,4}$ is demonstrated in Figure 2 for a square periodicity cell displayed in Figure 1.

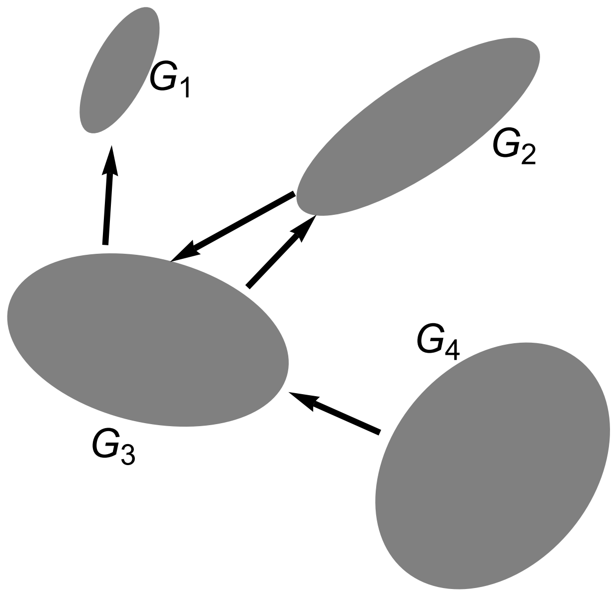

FIGURE 2. The element $U_{1,3,2,3,4}$ in G_1 emerges due to the sequence of impacts of the zero field in G_4 to G_3, the resulting field in G_3 to G_2, the resulting field in G_2 to G_3 and finally the resulting field in G_3 to G_1.

The potential in G_k is represented in the form of series equivalent to (5) for $|\varrho| < 1$ up to an additive constant c_k

$$u_k(\mathbf{x}) = u_0(\mathbf{x}) + \sum_{s=1}^{\infty} \varrho^s \sum_{k_1,k_2,\ldots,k_s} [U_{k_1,k_2,\ldots,k_s}(\mathbf{x}) - U_{k_1,k_2,\ldots,k_s}(\mathbf{w})] + c_k, \quad \mathbf{x} \in G_k \cup L_k \quad (k = 1, 2, \ldots, N), \tag{6}$$

where $\mathbf{w}$ is a fixed point outside of $\cup_k^N (G_k \cup L_k)$. The additional constant term $U_{k_1,k_2,\ldots,k_s}(\mathbf{w})$ is introduced in order to obtain the uniformly convergent series for a high dielectric constant of inclusions ($\varrho = 1$) and for insulating inclusions ($\varrho = -1$). The order of summation by levels is important for $|\varrho| = 1$ because the series (6) may be conditionally convergent [61, 26].

In the case of circular inclusions, the above scheme coincides with the image method based on the Kelvin transform. The elements of series (6) are constructed by the classic Schottky group of inversions on circles L_k [49]. The series (6) becomes the θ_2-Poincaré series which determines an authomorphic function [47]. Computationally effective methods for the Poincaré series were developed in [56] and other works.

The constructive applications of contrast expansion (5) in the theory of composites are usually accompanied by a comment on sufficiently small $|\varrho|$. Such an assumption for dispersed composites is redundant since the series (5) converges absolutely in the unit disk $|\varrho| < 1$ of the complex plane [60, 26, 58].

Schwarz's method is reduced to a system of integral equations whose kernels are Green's functions of separate inclusions occupying simply connected domains [58]. The similar Lippmann-Schwinger type equation [63] is stated over the total boundary of inclusions. Both methods ultimately lead to the same series (5). The methodology outlined in [63] is based on absolute convergence, i.e., on the Neumann series estimated by the norm of the integral operator. It turns out that the uniform convergence of Schwarz's method gives more general results than absolute convergence. It requires subtle construction of the conditionally convergent series (6) dependent on the order of summation.

We now define the class of *dispersed composite* First, every domain G_k or its convex hull is assumed to be a bounded convex set. Fix a domain G_k and describe its properties. Its boundary, closed curve L_k, can be locally parametrized by a $C^{(1,\alpha)}$ function except at a finite set of points where one-sided derivatives along L_k exist but can be different. Let a_k denote the gravitational center of the domain G_k. It will be convenient to consider a_k as a complex number

$$a_k = \frac{1}{|G_k|}\int_{G_k} z\, \mathrm{d}x_1 \mathrm{d}x_2 \quad (z = x_1 + \mathrm{i}x_2). \tag{7}$$

Introduce the generalized radius of G_k as $r_k = \sup_{z\in L_k} |z - a_k|$. Hereafter, the distance $|z-w|$ is considered in the plane torus topology. It is assumed that the closed domains $(G_k \cup L_k)$ for $k = 1,2,\dots,N$ are mutually disjoint and

$$r_k + r_m < |a_k - a_m|, \quad \text{for } k \neq m \quad (k,m = 1,2,\dots,N). \tag{8}$$

We call such a set $\{G_1, G_2, \dots, G_N\}$ by inclusions in a dispersed composite. The exterior domain G to the inclusions is called the host or matrix of the dispersed composite.

Introduce the maximum generalized radius $r = \max_{m=1,2,\dots,N} r_m$. It will be convenient to estimate the effective constants using the concentration of inclusions

$$f = \sum_{k=1}^{N} |G_k|, \tag{9}$$

where $|G_k|$ stands for the area of the domain G_k in the periodicity cell of the normalized unit area $|Q| = 1$. It follows from inequality $|G_k| \leq \pi r_k^2$ that

$$|G_k| = O(r^2), \quad \text{as } r \to 0 \quad \Leftrightarrow \quad N|G_k| = O(f), \quad \text{as } f \to 0. \tag{10}$$

Here, N is considered as a constant; the value $|G_k|$ is of order $O\left(\pi(\frac{r_k}{r})^2 r^2\right)$. Therefore, only two geometrical equivalent infinitesimal parameters, r^2 and f, can be considered in asymptotic formulas.

An important consequence of Schwarz's method is the decomposition theorem, which can be loosely formulated as follows

Theorem 1 (Decomposition theorem)**.** *The effective properties tensor $\boldsymbol{\varepsilon}_\perp$ can be represented as a linear combination of the pure geometrical parameters of inclusions with the coefficients depending on the local physical constants.*

Consider for simplicity the formulas (6) for the local field in the inclusion G_k. The tensor $\boldsymbol{\varepsilon}_\perp$ can be calculated by averaging the local field in inclusions per a representative volume element

(11)
$$\boldsymbol{\varepsilon}_\perp = \sum_{s=1}^{\infty} \varrho^s \sum_{k,k_1,k_2,\ldots,k_s} \mathbf{c}_{k,k_1,k_2,\ldots,k_s},$$

where the tensors $\mathbf{c}_{k,k_1,k_2,\ldots,k_s}$ describe the mutual interactions between inclusions and depend only on their concentration, location, and shapes; ϱ^s are the coefficients expressed only on the physical constants. Equation (11) for multi-phase composites become

(12)
$$\boldsymbol{\varepsilon}_\perp = \sum_{s=1}^{\infty} \sum_{k,k_1,k_2,\ldots,k_s} \varrho_k \varrho_{k_1} \cdots \varrho_{k_s} \mathbf{c}_{k,k_1,k_2,\ldots,k_s},$$

where ϱ_k stands for the contrast parameter of the materials occupied G_k and G.

The decomposition (12) emerges due to the contrast expansion of $\boldsymbol{\varepsilon}_\perp$. Its first constructive form was derived in [52, Section 4.2].

3. Mathematical modeling and determination of effective properties

The most popular analytical formulas for the effective permittivity of dispersed composites are the Maxwell-Garnett [42] and Brugemann [10] approximations, see equations (13). We now postpone formulas associated with the famous neutral Hashin-Shtrikman assemblage [12, 46] for coated composites and discuss analytical formulas for two-phase composites based on the self-consistent method (SCM). A typical modern review of various analytical formulas for the effective properties of composites, including elastic media, begins with a discussion of these formulas and their numerous extensions. Lots of items are laid out on the counter and offered for usage. Some contradict others; some violate the Hashin-Shtrikman bounds; some disturb the symmetry of tensors. However, everything becomes allowed if the magic word ”model” is used. There is no theoretical justification when creating a new model, giving way to empirical reasons. Experiments might confirm the model.

We will not discuss a rich assortment of models and narrow our focus to the specific two-phase 2D macroscopically isotropic composites in order to derive precision of various modifications of SCM such as effective medium approximation, mean-field, Mori-Tanaka methods, etc. It will be shown that some popular modifications of SCM are covered by the first-order approximation of Schwarz's method. This investigation explains plenty of illusory different formulas which are reduced to the same Maxwell-Garnett type estimation for dilute composites. It will be demonstrated that some higher-order extensions of SCM violate homogenization principles and lead to methodologically misleading results.

SCM has multiple meanings in different fields of science and engineering. It is related to the many-body problem in physics [77], assembly in statistical mechanics [18], molecular dynamics simulations [2, 43], biological fields and others [79, 29, 24]. We pay particular attention to self-consistent method in the theory of composites by Maxwell [41, p.365] developed by Hill [33], Beran [8], Kröner [38] and many others, and its relations to Schwarzs method outlined in the next sections of this note.

J.C. Maxwell proposed SCM in 1873 [41]. S.G. Mikhlin developed the alternating method of Schwarz in 1949 [45]. Mikhlin's result can be considered a mathematical embodiment of Maxwell's approach. We are interested in the Maxwell-Garnett approximation derived in 1904 [42] and suitable for the complex dielectric permittivity. It is worth noting that the same formula for real conductivity (permittivity) was known as the Clausius-Mossotti approximation [62] published in 1850. It coincides with Maxwell's approximation [41]. This formula arose in the works by Lorenz (1869) and Lorentz (1879) in another context[2]. Perhaps the universality of mathematical modeling was not clear in the XIX century. The historical notes [40, 46] and references therein clarify the series of the above formulas.

The main result of the present note consists of applying Schwarz's method to composites and deriving new higher-order approximations for the tensor $\boldsymbol{\varepsilon}_{\perp}$. In particular, we demonstrate that Maxwell's SCM and its modern modifications coincide with the first-order approximation of Schwarz's method.

An extensive theoretical review of SCMs for a dispersed composite can be found in [66, 67] where the term "grain composite" was used. After these works, many engineers implemented SCMs and obtained numerical estimations of the effective constants for various composites. However, the principal theoretical question of the limitations of SCMs was still open. The argumentation of SCM validity was based on the condition of dilute concentration in [41, 33, 38, 67, 66]. It was noted that applying SCM to non-dilute composites, i.e., to the second-order terms in f, may yield physically unacceptable predictions [17, 22].

The obtained result for a self-consistent approach in a 2D permittivity statement has to be expected in general consideration, in particular for 2D-3D conductive and elastic composites, the macroscopic viscosity of suspensions, and so forth [26, 21], due to the universality of mathematical modeling. Hence, the precision analysis for the 2D permittivity of the present

[2] James Clerk Maxwell and J.C. Maxwell Garnett are different persons, L. Lorenz and H.A. Lorentz, too.

book can be extended to general composites and porous media. Analytical formulas for the effective properties were derived in [26, 21, 58] in the form of power series in the concentration of inclusions f and contrast parameter ϱ. The series was truncated to polynomials in $f^{1/2}$ and ϱ with the exactly found coefficients symbolically depending on the location and shape of inclusions. Moreover, the order precision $O(|\varrho|^m f^{p/2})$ of the derived formulas was justified. Here, m denotes the number of iterations in Schwarz's method, and p is related to the concentration approximation used at every iteration.

The literature presents numerous formulas for effective properties, often viewed as perturbations or extensions of lower-order formulas in f and ϱ to higher-order terms, albeit frequently without justification. Some of these formulas, although formally distinct, describe composites under similar deterministic or probabilistic frameworks. For instance, an apparent contradiction between two different models concerning the effective permittivity of elliptical inclusions was clarified in [59]. Some researchers overlook inconsistencies in analytical formulas and instead rely on numerical comparisons of specific datasets.

This raises a fundamental question: why do some formulas applied to the same composite differ? Typically, the issue stems from confusing terminology, where the term model is incorrectly used in place of formula. It is crucial not to conflate a model with the resulting formula in exact sciences. If a mathematically defined problem yields two different answers, it does not imply the consideration of two distinct models. The correct research approach defines the model as a priori and conducts the mathematical study as a posteriori, not the reverse. This principle equally applies to probabilistic problems involving random composites, where the distribution of inclusions must be explicitly defined within the problem statement. Subsequently, the mathematical expectation of the effective constants, uniquely determined for a well-described class of random composites, must be calculated, as demonstrated in [65].

In numerous studies, algorithms simulate random composites incorporating random variables whose probability distributions are often obscured within machine codes. Furthermore, simulation outcomes can vary depending on the employed protocols [76, 37]. Such differences may lead to varying results for the effective constants, which correspond naturally to different spatial distributions of inclusions.

To the end of the above discussion, we have the exact coefficients in our formulas [26, 21] in ϱ and f. Hence, we can compare formulas from the previous publications with the asymptotic estimations in order to make conclusions concerning the validity of the formulas called "models." Such a comparison of the Maxwell-Garnett formula (Clausius-Mossotti approximation) for equal disks yields the asymptotic formula

$$\varepsilon_e = \frac{1+\varrho f}{1-\varrho f} + O(f^3), \tag{13}$$

where the tensor $\boldsymbol{\varepsilon}_\perp$ takes the form $\varepsilon_e I$, the scalar ε_e denotes the macroscopic permittivity, I the identity matrix. The exact precision up to $O(f^3)$ was established in [57].

The Hashin-Shtrikman bounds for 2D two-phase macroscopically isotropic composites for real normalized permittivity of components ε_1 and $\varepsilon = 1$ when $\varepsilon_1 > 1$ can be written in the form [31]

$$\varepsilon_{HS}^- \le \varepsilon_e \le \varepsilon_{HS}^+, \tag{14}$$

where

$$\varepsilon_{HS}^- = 1 + \frac{2f(\varepsilon_1 - 1)}{(1-f)(\varepsilon_1 - 1) + 2}, \quad \varepsilon_{HS}^+ = \varepsilon_1 + \frac{2(1-f)(1-\varepsilon_1)\varepsilon_1}{f(1-\varepsilon_1) + 2\varepsilon_1}. \tag{15}$$

The bounds (15) in terms of ϱ and f becomes

$$\varepsilon_{HS}^- = \frac{1+\varrho f}{1-\varrho f}, \quad \varepsilon_{HS}^+ = \frac{1+\varrho}{1-\varrho}\,\frac{1-\varrho(1-f)}{1+\varrho(1-f)}. \tag{16}$$

It is worth noting that ε_{HS}^- coincides with the Maxwell-Garnett approximation (13).

In this note, we show that Schwarz's method can be implemented in the form of different iterative schemes known in the theory of composites as the contrast and cluster expansions [46, 76, 26]. The contrast expansion for 2D permittivity problems leads to a power series in ϱ. The cluster expansion means a series in $f^{1/2}$. Frequently, the authors of SCMs do not care about the precision in ϱ and $f^{1/2}$, and derive, in the best case, the same formula asymptotically equivalent to the Maxwell-Garnett formula. The analysis of precision [57, 26, 21, 50, 54] demonstrate a discrepancy between the rigorously derived formulas for the effective permittivity of disks/spheres and some formulas obtained by SCM.

It was proved in [50, 54] that an extension of Maxwell's approach to a finite collection of inclusions (cluster) may give, at most, the effective properties of dilute clusters. A simple feature of an expansion of the effective permittivity in f was formulated in [57]. Any general formula for 2D two-phase macroscopically anisotropic dispersed composites which does not include the location of disks holds up to $O(f^2)$; for macroscopically isotropic composites up to $O(f^3)$. This criterion of lower order formula could be extended to multi-phase composites except at the Hashin-Shtrikman type assemblages [12, 15] with neutral inclusions. For instance, a set of valuable formulas for multi-phase macroscopically anisotropic composites [44] is valid up to $O(f^2)$.

4. Transmission conditions and $\mathbb{R}$-linear problem for 2D composites

We now proceed to implement Schwarz's method for boundary value problems of dispersed 2D periodic composites. Consider a multi-phase composite with non-overlapping inclusions embedded in the 2D matrix. The composite is supposed to be represented by a periodicity cell Q introduced as follows. The plane coordinates $\mathbf{x} = (x_1, x_2)$ will be identified with the complex variable $z = x_1 + \mathrm{i}x_2$. Let $\boldsymbol{\omega_1}$ and $\boldsymbol{\omega_2}$ be the fundamental periods on the complex plane $\mathbb{C}$

$$\boldsymbol{\omega_1}\mathrm{Im}\,\boldsymbol{\omega_2} = 1. \tag{17}$$

Equation (17) means that the area of Q is normalized to unity. Let the fundamental parallelogram Q

$$Q \equiv Q_{(0,0)} = \left\{ z = t_1\boldsymbol{\omega_1} + t_2\boldsymbol{\omega_2} \in \mathbb{C} : -\frac{1}{2} < t_1, t_2 < \frac{1}{2} \right\}. \tag{18}$$

The points $m_1\boldsymbol{\omega_1} + m_2\boldsymbol{\omega_2}$ $(m_1, m_2 \in \mathbb{Z})$ generate a doubly periodic lattice $\mathcal{Q}$ For instance, the square array is generated by two translation vectors expressed by complex numbers

$$\boldsymbol{\omega_1} = 1, \quad \boldsymbol{\omega_2} = \mathrm{i}. \tag{19}$$

Consider N non-overlapping simply connected domains G_k in the unit periodicity cell Q with piece-wise Lyapunov's boundaries L_k and the multiply connected domain $G = Q \setminus \cup_{k=1}^{N}$ $(G_k \cup L_k)$, the complement of all the closures of G_k to Q (see Figure 1). Each simple closed curve L_k leaves G_k to the left. Every curve L_k is smooth except at a finite set of points called vertices. Following [64], we say that a function belongs to Muskhelishvili's class H if it is Hölder continuous on all smooth closed arcs of L_k.

Consider a doubly periodic multi-phase composite when the host $G + m_1\boldsymbol{\omega_1} + m_2\boldsymbol{\omega_2}$ and the inclusions $G_k + m_1\boldsymbol{\omega_1} + m_2\boldsymbol{\omega_2}$ are occupied by dielectric materials. Let the permittivity of the host is normalized to unity, and the permittivity of kth inclusion be a complex number $\varepsilon_k = \varepsilon_k' + \mathrm{i}\varepsilon_k''$, where $\varepsilon_k' = \mathrm{Re}\,\varepsilon_k$ and $\varepsilon_k'' = \mathrm{Im}\,\varepsilon_k$. One can consider ε_k as the ratio of the permittivity of the kth inclusion to the permittivity of the matrix, where the dimension permittivities can be complex. One can assume that the constants ε_k take the values from a set $\mathcal{M}$ which contains less than N elements. Let $\#\mathcal{M} = M$, i.e., the composite is $(M+1)-$phases and $\varepsilon_k = \varepsilon^{(j)}$, if $j = 1, 2, \ldots, M$. In this case, formulas similar to the formulas (4.2.26)-(4.2.27) from [26] can be derived for a multi-phase composite by the method developed in [48]. These formulas can be considered as extensions of formulas [73] to higher order contrast parameters terms.

The external flux $\mathbf{E}_0$

Let $u = u' + iu''$ and $u_k = u_k' + iu_k''$ denote the potentials in G and G_k, respectively, where for instance $u' = \mathrm{Re}\,u$ and $u'' = \mathrm{Im}\,u$ in G. The complex functions u and u_k satisfy Laplace's equation in the corresponding domains and are continuously differentiable in their closures except at the vertices $W_k \subset L_k$ where they belong to Muskhelishvili's class H [64].

The perfect contact between the components is expressed by equations

$$u(t) = u_k(t), \quad \frac{\partial u}{\partial \mathbf{n}}(t) = \varepsilon_k \frac{\partial u_k}{\partial \mathbf{n}}(t), \quad t \in L_k \quad (k = 1, 2, \ldots, N), \tag{20}$$

where the normal derivative $\frac{\partial}{\partial \mathbf{n}}$ on L_k is used. Two complex relations (20) can be written in the extended real form

$$\begin{aligned} &u'(t) = u'_k(t), \quad \frac{\partial u'}{\partial \mathbf{n}}(t) = \varepsilon'_k \frac{\partial u'_k}{\partial \mathbf{n}}(t) - \varepsilon''_k \frac{\partial u''_k}{\partial \mathbf{n}}(t), \\ &u''(t) = u''_k(t), \quad \frac{\partial u''}{\partial \mathbf{n}}(t) = \varepsilon''_k \frac{\partial u'_k}{\partial \mathbf{n}}(t) + \varepsilon'_k \frac{\partial u''_k}{\partial \mathbf{n}}(t), \\ &\qquad t \in L_k \quad (k = 1, 2, \ldots, N). \end{aligned} \tag{21}$$

Following the homogenization theory [7], we must consider a composite in the plane torus topology. Hence, such a structure can be considered as a doubly periodic representative unit cell Q; see the MMM principle by Hashin [30] and its constructive application to random composites in Chapter 3 of [26].

The external field is modeled by two conditions

$$u(z + \boldsymbol{\omega_1}) - u(z) = \xi_1, \quad u(z + \boldsymbol{\omega_2}) - u(z) = \xi_2, \tag{22}$$

where ξ_1 and ξ_2 are two complex vectors that determine the jump of the external field in the directions $\boldsymbol{\omega_1}$ and $\boldsymbol{\omega_2}$, respectively. It follows from the theory of homogenization [7] that the external potential $u_{ext}(z)$ has to be taken as a linear function in x_1 and x_2. A linear function in x_1 and x_2 can be represented as an $\mathbb{R}$-linear function in the complex variable $z = x_1 + \mathrm{i}x_2$

$$u_{ext}(z) = \mathbf{a}z + \mathbf{b}\overline{z}, \tag{23}$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.

The theory of homogenization implies that the problem (21)-(22) has to be solved for two linearly independent vectors ξ_1 and ξ_2 in order to completely determine the effective permittivity tensor. One can consider the normalized external fields parallel to the coordinate axes determined by the external potential (23) with $\mathbf{a} = \mathbf{b} = (\frac{1}{2}, 0)^\top$ and $\mathbf{a} = -\mathbf{b} = (\frac{1}{2\mathrm{i}}, 0)^\top$, respectively, where $^\top$ denotes the transposition. These fields yield two pairs

$$\xi_1 = (\boldsymbol{\omega_1}, 0)^\top, \ \xi_2 = (\mathrm{Re}\, \boldsymbol{\omega_2}, 0)^\top \quad \text{and} \quad \xi_1 = (0, 0)^\top, \ \xi_2 = (0, \mathrm{Im}\, \boldsymbol{\omega_2})^\top. \tag{24}$$

The corresponding calculations can be found in [26, Section 2.2] for the real fields.

In the present note and most others, we consider the square array with (19) for definiteness. In this case, the function $u(t)$ satisfies the normalized jump conditions per unit periodic square cell Q

$$u(z+1)-u(z)=(1,0)^\top, \quad u(z+i)-u(z)=(0,0)^\top. \tag{25}$$

The conditions (25) mean that the external complex flux $\mathbf{E}_0=(-1,0)$ is applied. Here, the components of $\mathbf{E}_0$ are complex numbers. More precisely, for instance, the real part of first relation (25) are formally written in the same form

$$\operatorname{Re} u(z+1)-\operatorname{Re} u(z)=(1,0)^\top, \quad \operatorname{Im} u(z+1)-\operatorname{Im} u(z)=(0,0)^\top. \tag{26}$$

Analogous relations occur for the second problem when $\mathbf{E}_0=(0,-1)$.

We now reduce the problem (21) to a vector-matrix $\mathbb{R}$-linear problem. The second column of (21) suggests to introduce the non-degenerate real matrix

$$\boldsymbol{\alpha}_k=\begin{pmatrix} \varepsilon_k' & -\varepsilon_k'' \\ \varepsilon_k'' & \varepsilon_k' \end{pmatrix}. \tag{27}$$

A similar matrix was introduced in [14] to develop variational principles for the complex effective tensor.

Introduce the vector complex potentials

$$\boldsymbol{\varphi}(z)=\begin{pmatrix} u'(z)+iv'(z) \\ u''(z)+iv''(z) \end{pmatrix}, \quad z\in G \tag{28}$$

and

$$\boldsymbol{\varphi}_k(z)=\frac{1}{2}(I+\boldsymbol{\alpha}_k)\begin{pmatrix} u_k'(z)+iv_k'(z) \\ u_k''(z)+iv_k''(z) \end{pmatrix}, \quad z\in G_k. \tag{29}$$

where $v'(z)$, $v''(z)$ and $v_k'(z)$, $v_k''(z)$ denote the imaginary parts of the components of the analytic (meromorphic) vector-functions $\boldsymbol{\varphi}(z)$ and $\boldsymbol{\varphi}_k(z)$, respectively, I denotes the identity matrix.

The harmonic and analytic functions are related by the vector relations The complex vector flux

$$\boldsymbol{\psi}(z)=\begin{pmatrix} \frac{\partial u'}{\partial x_1}-\mathrm{i}\frac{\partial u'}{\partial x_2} \\ \\ \frac{\partial u''}{\partial x_1}-\mathrm{i}\frac{\partial u''}{\partial x_2} \end{pmatrix}, \quad z\in G, \tag{30}$$

and

$$\boldsymbol{\psi}_k(z)=\frac{1}{2}(I+\boldsymbol{\alpha}_k)\begin{pmatrix} \frac{\partial u_k'}{\partial x_1}-\mathrm{i}\frac{\partial u_k'}{\partial x_2} \\ \\ \frac{\partial u_k''}{\partial x_1}-\mathrm{i}\frac{\partial u_k''}{\partial x_2} \end{pmatrix}, \quad z\in G_k. \tag{31}$$

Using the formula

$$2\,(I+\boldsymbol{\alpha}_k)^{-1} = \frac{2}{|1+\varepsilon_k|^2}\left(I+\boldsymbol{\alpha}_k^T\right) = \frac{2}{|1+\varepsilon_k|^2}\begin{pmatrix} 1+\varepsilon_k' & \varepsilon_k'' \\ -\varepsilon_k'' & 1+\varepsilon_k' \end{pmatrix}, \tag{32}$$

introduce the contrast matrix parameter

$$\boldsymbol{\beta}_k = -(I-\boldsymbol{\alpha}_k)(I+\boldsymbol{\alpha}_k)^{-1} = \frac{1}{|1+\varepsilon_k|^2}\begin{pmatrix} |\varepsilon_k|^2-1 & -2\varepsilon_k'' \\ 2\varepsilon_k'' & |\varepsilon_k|^2-1 \end{pmatrix}. \tag{33}$$

Then, the conditions (21) can be written in the form of vector-matrix $\mathbb{R}$-linear problem

$$\boldsymbol{\varphi}(t) = \boldsymbol{\varphi}_k(t) - \boldsymbol{\beta}_k\overline{\boldsymbol{\varphi}_k(t)}, \quad t \in L_k \quad (k = 1,2,\ldots,N). \tag{34}$$

It follows from (25) that the vector-function $\boldsymbol{\varphi}(z)$ satisfies the quasi-periodicity conditions

$$\boldsymbol{\varphi}(z+1) = \boldsymbol{\varphi}(z) + \begin{pmatrix} 1+\mathrm{i}d_{11} \\ \mathrm{i}d_{21} \end{pmatrix}, \ \boldsymbol{\varphi}(z+i) = \boldsymbol{\varphi}(z) + \begin{pmatrix} \mathrm{i}d_{12} \\ \mathrm{i}d_{22} \end{pmatrix}, \tag{35}$$

where d_{ij} are undetermined real constants that could be found during the solution to the boundary value problem (34).

The eigenvalues of the matrix $\boldsymbol{\beta}_k$

$$\varrho_k = \frac{\varepsilon_k - 1}{\varepsilon_k + 1}, \quad \overline{\varrho_k} = \frac{\overline{\varepsilon_k} - 1}{\overline{\varepsilon_k} + 1}. \tag{36}$$

The matrix $\boldsymbol{\beta}_k$ admits the decomposition

$$\boldsymbol{\beta}_k = \mathbf{T}\Lambda_k\mathbf{T}^{-1}, \tag{37}$$

where

$$\mathbf{T} = \begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix}, \quad \mathbf{T}^{-1} = \frac{1}{2}\begin{pmatrix} -\mathrm{i} & 1 \\ \mathrm{i} & 1 \end{pmatrix}, \quad \Lambda_k = \begin{pmatrix} \varrho_k & 0 \\ 0 & \overline{\varrho_k} \end{pmatrix}. \tag{38}$$

Equation (37) implies that

$$\boldsymbol{\beta}_k\boldsymbol{\beta}_m = \mathbf{T}\begin{pmatrix} \varrho_k\varrho_m & 0 \\ 0 & \overline{\varrho_k}\,\overline{\varrho_m} \end{pmatrix}\mathbf{T}^{-1}. \tag{39}$$

In the case of real ε_k, the matrix (33) becomes diagonal $\boldsymbol{\beta}_k = \varrho_k I$, and the vector-matrix $\mathbb{R}$-linear problem (34) can be decomposed onto two the same scalar $\mathbb{R}$-linear problems systematically studied in [26]. The contrast expansion method equivalent to Schwarz's method can be applied to the vector-matrix problem to determine the local fields. It is suggested that the convergence will hold for $|\varrho_k| \le 1$.

In the subsequent sections of this note, we assume that permittivity takes real values. Consequently, the vector complex potentials mentioned earlier share two identical coordinates. This enables us to reduce vector-matrix operations to scalar transformations, as elaborated in [26]. We will proceed with our analysis under the assumption of real permittivity, deferring the exploration of the complex case.

5. Schwarz's method for dispersed periodic composites with real-valued permittivity

It is noted at the end of the previous section that Schwarz's method can be applied to the vector-matrix $\mathbb{R}$-linear problem (34). As a result, the local fields can be found by power series in the matrices $\boldsymbol{\beta}_k$, perhaps truncated, but in symbolic form. Such an expression can be helpful in the computation of the effective permittivity tensor $\boldsymbol{\varepsilon}_\perp$. However, we will discover that solving the scalar $\mathbb{R}$-linear problem is adequate after the application of the general theorem of analytic continuation to the complex domain of ε_k in formulas for the tensor $\boldsymbol{\varepsilon}_\perp$ with real ε_k. Therefore, we can adopt the following strategy. First, we obtain the symbolic form of $\boldsymbol{\varepsilon}_\perp$ by considering real ε_k and solving the scalar $\mathbb{R}$-linear problem discussed in this note. This step alone yields the desired result. Next, we extend the obtained analytical exact and approximate formulas by formally considering complex values of ε_k. This approach simplifies matrix transformations and reduces them to more manageable scalar manipulations.

In addition, we will offer a lucid illustration of the methodological limitations of EMA in the scalar case. This demonstration will be comprehensively presented towards the note's conclusion, following our rigorous investigation of Schwarz's method.

So, hereafter non-overlapping inclusions G_k of real permittivity ε_k $(k = 1, 2, \ldots, N)$ are embedded in a periodic square cell Q of unit area. The following equations establish a relationship between the scalar complex and real potentials.

$$\text{Re}\,\varphi(z) = u(z),\ z \in G,\ \text{Re}\,\varphi_k(z) = \frac{\varepsilon_k + 1}{2} u_k(z),\ z \in G_k\ (k = 1, 2, \ldots, N). \tag{40}$$

The complex potentials $\varphi(z)$ and $\varphi_k(z)$ are analytic in G and G_k, respectively, and continuously differentiable in the closures of the considered domains except at the vertices of L_k where the limit values of derivatives $\varphi'(z)$ and $\varphi'_k(z)$ belong to Muskhelishvili's class H. The function $\varphi(z)$ satisfies the quasi-periodicity conditions

$$\varphi(z+1) = \varphi(z) + 1,\ \varphi(z+i) = \varphi(z) + id, \tag{41}$$

d is an undetermined real constant that should be found while solving the problem [26, Chapter 3, Section 4.3]. The perfect contact between the components (transmission condition) is written as the scalar $\mathbb{R}$-linear problem

$$\varphi(t) = \varphi_k(t) - \varrho_k \overline{\varphi_k(t)}, \quad t \in L_k \quad (k = 1, 2, \ldots, N). \tag{42}$$

Introduce a space $\mathcal{H}(G^+)$ in $G^+ = \cup_{k=1}^n G_k$ and Hölder continuous in the closure of G^+ endowed the norm [9]

$$(43)\qquad ||\omega|| = \sup_{t\in L}|\omega(t)| + \sup_{t_{1,2}\in L}\frac{|\omega(t_1)| - \omega(t_2)|}{|t_1-t_2|^\alpha},$$

where $0 < \alpha \leq 1$ and $L = \cup_{k=1}^N L_k$. The space $\mathcal{H}(G_k)$ of functions analytic in a fixed G_k and Hölder continuous in $G_k \cup L_k$ can also be considered. The problem (41)-(42) is considered in the space $\mathcal{H}(G^+)$ [61].

Let $\widehat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$ denote the extended complex plane. Let $h(t)$ be a function Hölder continuous on L. Consider the Cauchy-type integral on the complex plane [64]

$$(44)\qquad \Phi(z) = \frac{1}{2\pi i}\int_L \frac{h(t)}{t-z}\mathrm{d}t, \quad z \in \widehat{\mathbb{C}}\backslash L.$$

Its generalization to a class of doubly periodic functions (on the plane torus) has the form [28]

$$(45)\qquad \Phi(z) = \frac{1}{2\pi i}\int_L h(t)E_1(t-z)\,\mathrm{d}t,\ z \in G^+\cup D.$$

It is defined by the Eisenstein function $E_1(z)$ expressed through the Weierstrass function $\zeta(z)$. The function (45) is double periodic up to constant jumps per the periodicity cell

$$(46)\qquad \Phi(z+\boldsymbol{\omega_j}) - \Phi(z) = -\frac{\delta_j}{\pi i}\int_L h(t)\,\mathrm{d}t \quad (j=1,2).$$

where δ_j are constants determined in [1].

Let the domain G^- be the complement of the closure of all the inclusion domains to the extended complex plane, i.e., $G^- = \widehat{\mathbb{C}}\backslash(G^+\cup L)$. If $h(z)$ belongs to $\mathcal{H}(G^+)$, according to Chauchy's formula, we have [23, 64]

$$(47)\qquad \frac{1}{2\pi i}\int_L \frac{h(t)}{t-z}\mathrm{d}t = \begin{cases} h(z), & z\in G^+, \\ 0, & z \in G^-. \end{cases}$$

If $h(z)$ belongs to $\mathcal{H}(G^-)$, we have [23, 64]

$$(48)\qquad \frac{1}{2\pi i}\int_L \frac{h(t)}{t-z}\mathrm{d}t = \begin{cases} h(\infty), & z\in G^+, \\ -h(z)+h(\infty), & z \in G^-. \end{cases}$$

The same formulas (47)-(48) hold for the Cauchy's integral (45) with the other kernel on the torus topology with $G^- = G$.

The limit boundary values $\Phi^+(t) = \lim_{G_k\ni z\to t}\Phi(z)$ and $\Phi^-(t) = \lim_{D\ni z\to t}\Phi(z)$ of the Cauchy-type integral (44) satisfy Sochocki's formulas

$$(49)\qquad \Phi^+(t) = \frac{1}{2}h(t) + \frac{1}{2\pi i}\int_L \frac{h(\tau)}{\tau - t}\mathrm{d}\tau,\ \Phi^-(t) = -\frac{1}{2}h(t) + \frac{1}{2\pi i}\int_L \frac{h(\tau)}{\tau - t}\mathrm{d}\tau,\ t\in L_k.$$

The singular integral $\frac{1}{2\pi i}\int_L \frac{h(\tau)}{\tau - t}\mathrm{d}\tau$ from (49) is defined as the principal value integral [23, 64]. The same formulas hold for (45) with the other kernel

$$\Phi^+(t) = \tfrac{1}{2}h(t) + \tfrac{1}{2\pi i}\int_L h(\tau)E_1(\tau - t)\mathrm{d}\tau,$$
(50)
$$\Phi^-(t) = -\tfrac{1}{2}h(t) + \tfrac{1}{2\pi i}\int_L h(\tau)E_1(\tau - t)\mathrm{d}\tau,\ t \in L_k.$$

Application of the Cauchy-type integral (45) to (42) yields the following system of integral equations [26]

$$\varphi_k(z) = \sum_{m=1}^{N} \frac{1}{2\pi i}\int_{L_m} \varrho_m \overline{\varphi_m(t)} E_1(t - z)\, \mathrm{d}t + z + c_k,\ z \in G_k\ (k = 1, 2, \ldots, N), \tag{51}$$

where c_k are undetermined constants. Equation (51) can be considered as an explicit form of the general equation similar to (3)

$$u_k = \sum_{m=1}^{N} \varrho_m F_m u_m + u_0, \quad z \in G_k\ (k = 1, 2, \ldots, N). \tag{52}$$

This equation expresses the charge (energetic) balance between the field u_k in the inclusion G_k from one side and the external field u_0 and the fields $F_m u_m$ induced by the inclusions G_m from the other side. Schwarzs method can be considered as an iterative scheme applied to equations (51) or to (52).

Koiter developed the theory of singular integral equations for doubly periodic problems [36], see also [16] and historical notes in [28, p.55-58]. The theory of singular integral equations for doubly periodic problems was further developed by Filshtinsky [28, 53] who used the Weierstrass function $\zeta(z)$ instead of the Eisenstein function $E_1(z)$. It was established in [9, 26] that the system (51) for $|\varrho_m| \le 1$ with fixed c_k has a unique solution in $\mathcal{H}(G^+)$. This solution can be found by the method of successive approximations converging in $\mathcal{H}(G^+)$, i.e., uniformly in $\cup_{k=1}^{N}(G_k \cup L_k)$.

When the functions $\varphi_k(z)$ are determined, the complex potential in the domain G is calculated by the formula [26]

$$\varphi(z) = \sum_{m=1}^{N} \frac{\varrho_k}{2\pi i}\int_{L_m} \overline{\varphi_m(t)} E_1(t - z)\, \mathrm{d}t + z + c_k,\ z \in G. \tag{53}$$

Using Sochocki's formulas (50) one can subtract the limit values of (53) from (51) on L_k in order to check the relation $\varphi_k(t) - \varphi(t) = \varrho\, \overline{\varphi_k(t)}$, $t \in L_k$, equivalent to (42).

The normalized effective permittivity tensor in the considered case of real ε_k can be calculated by the formula (3.2.42) of the book [26]. Two components of the tensor are given in the form convenient in the further consideration of complex ε_k

$$\varepsilon_{11} = 1 + 2\mathrm{Re}\ \sum_{k=1}^{N} \varrho_k \int_{G_k} \frac{d\varphi_k}{dz}(\xi_1 + i\xi_2)\ \mathrm{d}\xi_1 \mathrm{d}\xi_2,$$

(54)

$$\varepsilon_{12} = -2\mathrm{Im}\ \sum_{k=1}^{N} \varrho_k \int_{G_k} \frac{d\varphi_k}{dz}(\xi_1 + i\xi_2)\ \mathrm{d}\xi_1 \mathrm{d}\xi_2.$$

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